1

Max-flow extension:

For a directed $G=(V,E)$ with capacities $c(e)>0$ for $e\in E$ & demands $d(e)\geq 0$ for $e\in E$

A feasible flow is a flow where

for $e\in E$, $d(e)\leq f(e)\leq c(e)$.

Is there a feasible flow?

Reduce to max-flow.

Define $G'=(V',E')$ as:

Vertices: keep V and add vertices s & t

Edges: for $e\in E$, add e to G' with new $c'(e)=c(e)-d(e)$.

for $v\in V$, add $\overrightarrow{sv}$ with $c'(\overrightarrow{sv})=d^{in}(v)=\sum_{u} d(\overrightarrow{uv})$

add $\overrightarrow{vt}$ with $c'(\overrightarrow{vt})=d^{out}(v)$

add $t\to s$ with $c'(\overrightarrow{ts})=\infty$.

Let $D = \sum_{e \in E} \ell(e)$

Note $c^{out}(s') = D$ & $c^{in}(t') = D$

Thus ~~size(f')~~ for a flow f' in G',

$$\text{size}(f') \leq D.$$

Say f' is <u>saturating</u> if $\text{size}(f') = D$.

<u>Lemma</u>: G has a feasible flow $\iff$ G' has a saturating flow.

This gives a scheme to find a feasible flow. Run max-flow on G' & check if its size is $= D$. If so, use this max flow f' to get a feasible f for G. Then can use f to get a max-size feasible flow.

Proof:

$\Leftarrow$: Take f' in G' which is saturating

$$\text{Let } f(e) = f'(e) + l(e)$$

Now let's check that f is a valid flow in G.
& is feasible.

Need to verify that $f^{in}(v) = f^{out}(v)$ for $v \in V-s-t$.

For $v \in V-s-t$, we know $f'^{in}(v) = f'^{out}(v)$

$$f'^{in}(v) = \sum_{u \in V} f'(\overrightarrow{uv}) + f'(\overrightarrow{s'v})$$

$$= \sum_{u} f'(\overrightarrow{uv}) + l^{in}(v)$$

$$f'^{out}(v) = \sum_{w} f'(\overrightarrow{vw}) + l^{out}(v)$$

$$f^{in}(v) = \sum_{u \in V} f'(\overrightarrow{uv}) + l(\overrightarrow{uv}) = f'^{in}(v)$$

Similarly, $f^{out}(v) = f'^{out}(v)$

& hence, $f^{in}(v) = f^{out}(v)$.

Note also $0 \leq f'(e) \leq c'(e) = c(e) - l(e)$

hence $l(e) \leq f(e) \leq c(e)$ ∎

$\Rightarrow$: Given a feasible f let's construct a saturating flow f' for G':

for $e \in E$, let $f'(e) = f(e) - d(e)$.

for $v \in V$, let $f'(\overrightarrow{s'v}) = d^{in}(v)$

$$f'(\overrightarrow{vt'}) = d^{out}(v)$$

$$f'(t \to s) = \text{size}(f).$$

Note: $f(e) \geq d(e)$ so $f'(e) \geq 0$

$f(e) \leq c(e)$ so $f'(e) \leq c(e) - d(e) \leq c'(e)$.

$$f'^{in}(v) = d^{in}(v) + f^{in}(v) - d^{in}(v) = f^{in}(v)$$

$$f'^{out}(v) = f^{out}(v)$$

hence, $f'^{in}(v) = f'^{out}(v)$. $\boxtimes$

How do we find a maximum size feasible flow?

Find a feasible flow using G'.

Then run Max-flow alg. to augment but residual graph G^f as:

$$c_f(\overrightarrow{vw}) = \begin{cases} c(\overrightarrow{vw}) - f(\overrightarrow{vw}) & \overrightarrow{vw} \in E \\ f(\overrightarrow{wv}) - \underbrace{d(\overrightarrow{wv})}_{\text{new part.}} & \overrightarrow{wv} \in E \\ 0 & \text{otherwise} \end{cases}$$

new part.
Since can't reduce flow below $d(\overrightarrow{wv})$.

But can do easier using linear programming.