LP = linear programming

optimization problems with linear objective function & linear constraints.

Simple example:

Company makes A & B
How much of each to produce to maximize profit?
Each unit of A makes a profit of \$1
& each of B makes \$6.
Demand: ≤ 300 units of A/day & ≤ 200 of B.

Supply: ≤ 700 hours/day, where
A takes 1 hour/unit & B takes 3/unit.

Variables: x_1 & x_2 where x_i = # of units of item i produced per day

associated LP: objective function: $\max\ x_1 + 6x_2$

constraints:

$x_1 \leq 300$ ①

$x_2 \leq 200$ ②

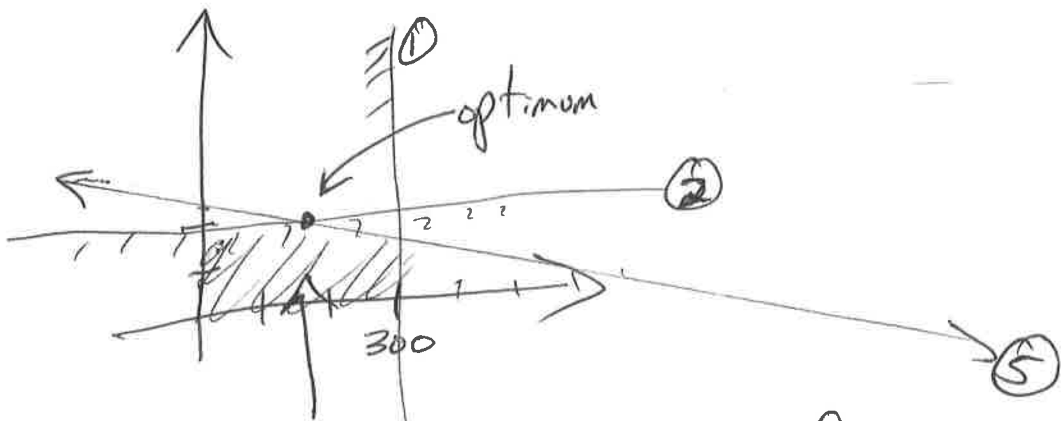
$x_1 \geq 0$ ③

$x_2 \geq 0$ ④

$x_1 + 3x_2 \leq 700$ ⑤

Allow x_i to take fractional amount.

5 constraints = linear inequality = half-space
in 2 dimensions (since 2 variables)



feasible region = valid x

Goal: find max c where $x_1 + 6x_2 = c$
intersects feasible region

optimum is $c = 1300$
from $(x_1, x_2) = (100, 200)$

optimum happens to occur at integer values,
but may not always do so.

Standard form for LP's:

Variables $x_1, \ldots, x_n$

$$\max \quad c_1 x_1 + \cdots + c_n x_n$$

$$\left.\begin{aligned} \text{s.t.} \quad a_{11}x_1 + \cdots + a_{1n}x_n &\le b_1 \\ &\vdots \\ a_{n1}x_1 + \cdots + a_{mn}x_n &\le b_m \end{aligned}\right\} m \text{ constraints}$$

$$x_1, x_2, \ldots, x_n \ge 0$$

In linear algebra form:

Variable vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

objective function $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$m \times n$ constraint matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \vdots & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

standard/generic form for LP's:

$$\max \ c^T x$$
$$Ax \le b$$
$$x \ge 0$$

How to convert into standard form?

$$\min c^T x \iff \max -c^T x$$

$$a_1x_1 + \cdots + a_nx_n \geq b \iff -a_1x_1 - \cdots - a_nx_n \leq -b$$

$$a_1x_1 + \cdots + a_nx_n = b \iff a_1x_1 + \cdots + a_nx_n \leq b$$
$$\& \ a_1x_1 + \cdots + a_nx_n \geq b$$

Can't use strict inequalities:

For example: $\max x$
$\quad$ s.t. $x < 100$

what's the solution?

for variable x_i without lower bound $(x_i \geq 0)$
make 2 new variables x_i^+ & x_i^-
& replace $x_i = x_i^+ - x_i^-$ where $x_i^+ \geq 0$
& $x_i^- \geq 0$

Algorithms for solving LP's:

Polynomial-time algorithms using:
Ellipsoid method & interior-point methods

Simplex algorithm: widely-used on HUGE LP's
but worst case running time is exponential.

Another example:

3 products A, B, C

Profit: \$1/unit of A, \$6 per B, \$10 per C

Demand: ≤ 300 units/day of A, ≤ 200 for B

Supply: ≤ 1000 hours total/day,

A takes 1 hour, B takes 3, C takes 2.

Packaging: ≤ 500 units total/day

B takes 1, & C takes 3.

LP:

$$\max \; x_1 + 6x_2 + 10x_3$$

$$\text{s.t.} \quad x_1 \leq 300 \qquad (1)$$

$$x_2 \leq 200 \qquad (2)$$

$$x_1 + 3x_2 + 2x_3 \leq 1000 \qquad (3)$$

$$x_2 + 3x_3 \leq 500 \qquad (4)$$

$$x_1, x_2, x_3 \geq 0 \qquad (5a, 5b, 5c)$$

Simplex: Start at $\overline{x} = (0, \ldots, 0)$

Look for neighboring vertex that has higher objective value then move there

Repeat until better than all neighbors

n variables $\Rightarrow$ n dimensions

$n+m$ constraints

feasible region = convex polyhedron

= intersection of $n+m$ halfspaces

Vertices = corners of

= points satisfying n constraints with equality & remaining m with inequalities

$\leq \binom{m+n}{n}$ vertices

Vertex defined by n tight constraints

neighboring vertex: replace one of the n by another of the m.

$O(nm)$ neighboring vertices.

Simplex on example 2:

Start at (0,0,0) Profit = 0
vertex defined by 5a, 5b, 5c

↓

vertex defined by 1, 5b, 5c
(300, 0, 0) Profit = 300

↓

Vertex from 1, 2, 5c
(300, 200, 0) Profit = 1500

↓

Vertex from 1, 2, 3
(300, 200, 50) Profit = 2000

↓

Vertex from 2, 3, 4
(200, 200, 100) Profit = 2400

How do we verify it's optimal?

Let $y = (y_1, y_2, y_3, y_4) = (0, \frac{1}{3}, 1, \frac{8}{3})$

Look at $y_1 \times ① + y_2 \times ② + y_3 \times ③ + y_4 \times ④$

$$
\begin{array}{llr}
y_1 \times ① = 0\,(& x_1 & \leq 300) \\
y_2 \times ② = \frac{1}{3}\,(& x_2 & \leq 200) \\
y_3 \times ③ = 1\,(& x_1 + 3x_2 + 2x_3 & \leq 1000) \\
y_4 \times ④ = \frac{8}{3}\,(& x_2 + 3x_3 & \leq 500) \\
\hline
 & \underbrace{x_1 + 6x_2 + 10x_3}_{\text{Profit}} & \leq 2400
\end{array}
$$

So profit ≤ 2400

& hence $(200, 200, 100)$ is optimal.

How did we find the y to verify that example 2 is optimal at $(200, 200, 100)$?

Look at $y_1 \times ① + y_2 \times ② + y_3 \times ③ + y_4 \times ④$

plugging in ①, ②, ③, ④ we have:

$$x_1 y_1 + x_2 y_2 + x_1 y_3 + 3x_2 y_3 + 2x_3 y_3 + x_2 y_4 + 3x_3 y_4$$
$$\leq 300y_1 + 200y_2 + 1000y_3 + 500y_4$$

Rearrange LHS:

$$x_1(y_1 + y_3) + x_2(y_2 + 3y_3 + y_4) + x_3(2y_3 + 3y_4)$$

objective function is $x_1 + 6x_2 + 10x_3$

hence we'd like:

$$y_1 + y_3 \geq 1$$
$$y_2 + 3y_3 + y_4 \geq 6$$
$$2y_3 + 3y_4 \geq 10$$

& we want to minimize RHS

So dual LP: min $300y_1 + 200y_2 + 1000y_3 + 500y_4$

s.t. $y_1 + y_3 \geq 1$

$y_2 + 3y_3 + y_4 \geq 6$

$2y_3 + 3y_4 \geq 10$

Max-flow via LP:

Variables f_e for every $e \in E$.

flow-out of s

objective function: $\max \sum_{\vec{sz} \in E} f_{sz}$

Constraints:

for every $e \in E$: $0 \le f_e \le c_e$

for every $v \in V - \{s, t\}$: $\sum_{\vec{wv} \in E} f_{wv} = \sum_{\vec{vz} \in E} f_{vz}$