Directed Hamiltonian Path Problem

Input: $G = (V, E)$, a directed graph

$s, t \in V$

Problem: Is there a directed path from s to t in G that visits each node exactly once.

Traveling Salesman Problem

Input:
- $G = (V, E)$, a complete graph
- non-negative integer costs between every pair of vertices
- a budget b

Problem: Is there a cycle, that passes through every vertex exactly once, of cost $\leq b$?

DHP is NP-Complete

1 DHP is in NP

Given a candidate solution we can check if it is a Hamiltonian path in polynomial time.

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2 $3SAT \leq_p DHP$

- Let $F = C_1 \wedge C_2 \wedge \ldots \wedge C_m$ be a 3CNF formula where

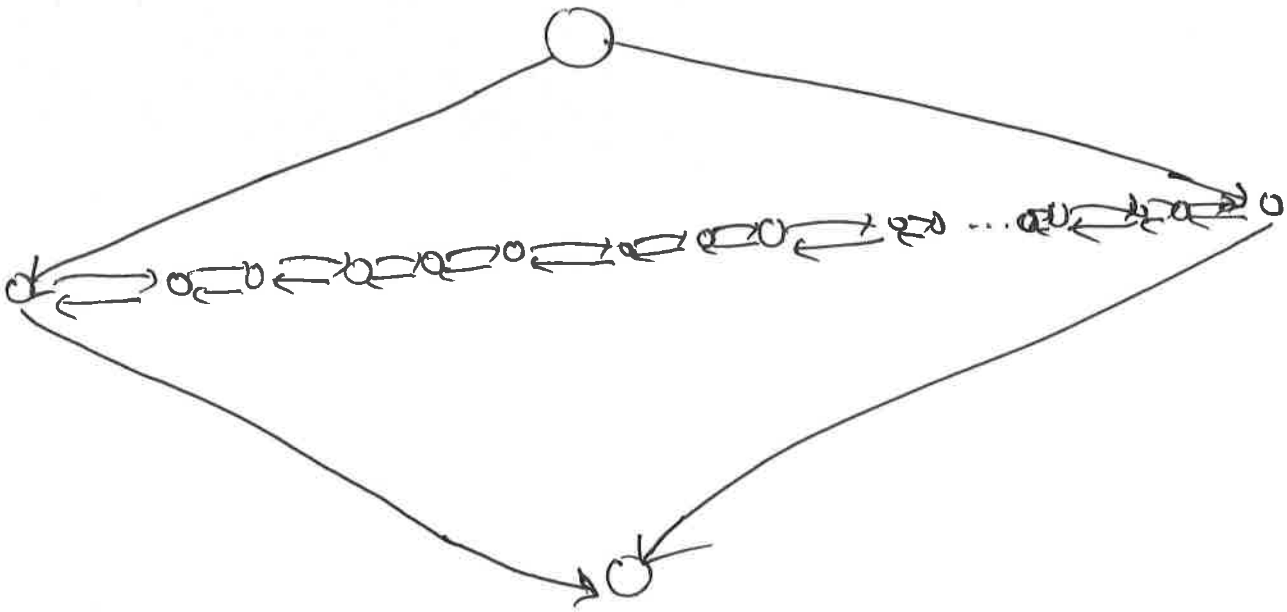
$$C_i = a_i \vee b_i \vee c_i$$

Here each a_i, b_i, c_i is a literal x_i or $\bar{x}_i$.

Let $x_1, \ldots, x_n$ be the n variables of F.

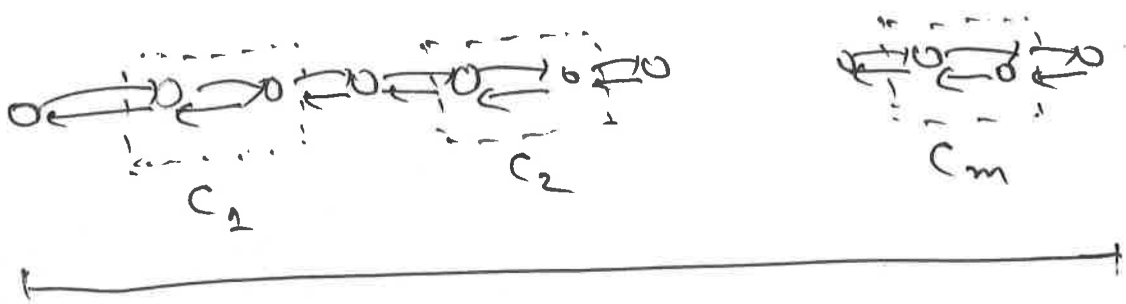
- Construct a directed graph $G = (V, E)$:

For each x_i construct a diamond-shaped graph structure (gadget) as follows:



Horizontal row of nodes:

- An adjacent pair of nodes for each clause
- Separator nodes next to the pairs:



$3m+1$ nodes.

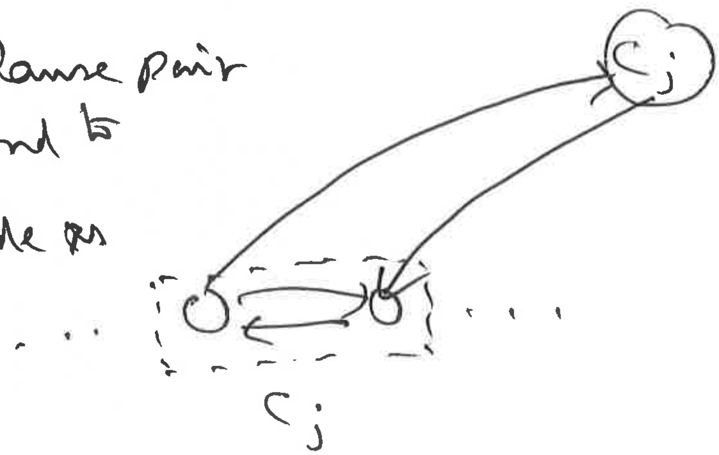
(m clause pairs)

For each clause C_j there is a single node:

Connecting the variable and clause gadgets

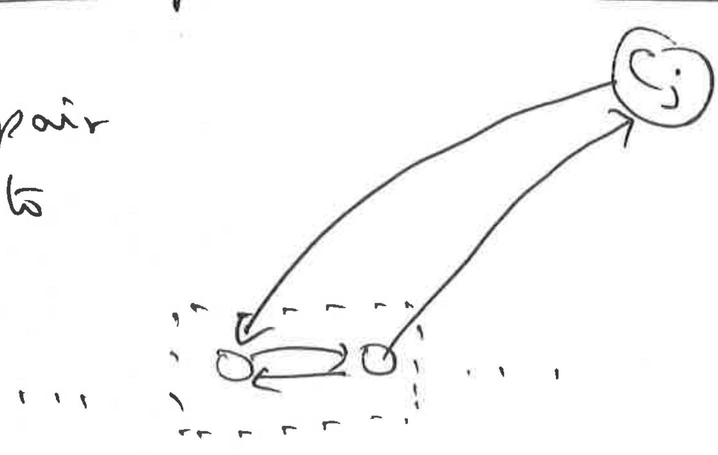
Case 1: Variable x_j appears as a +ve literal in C_j

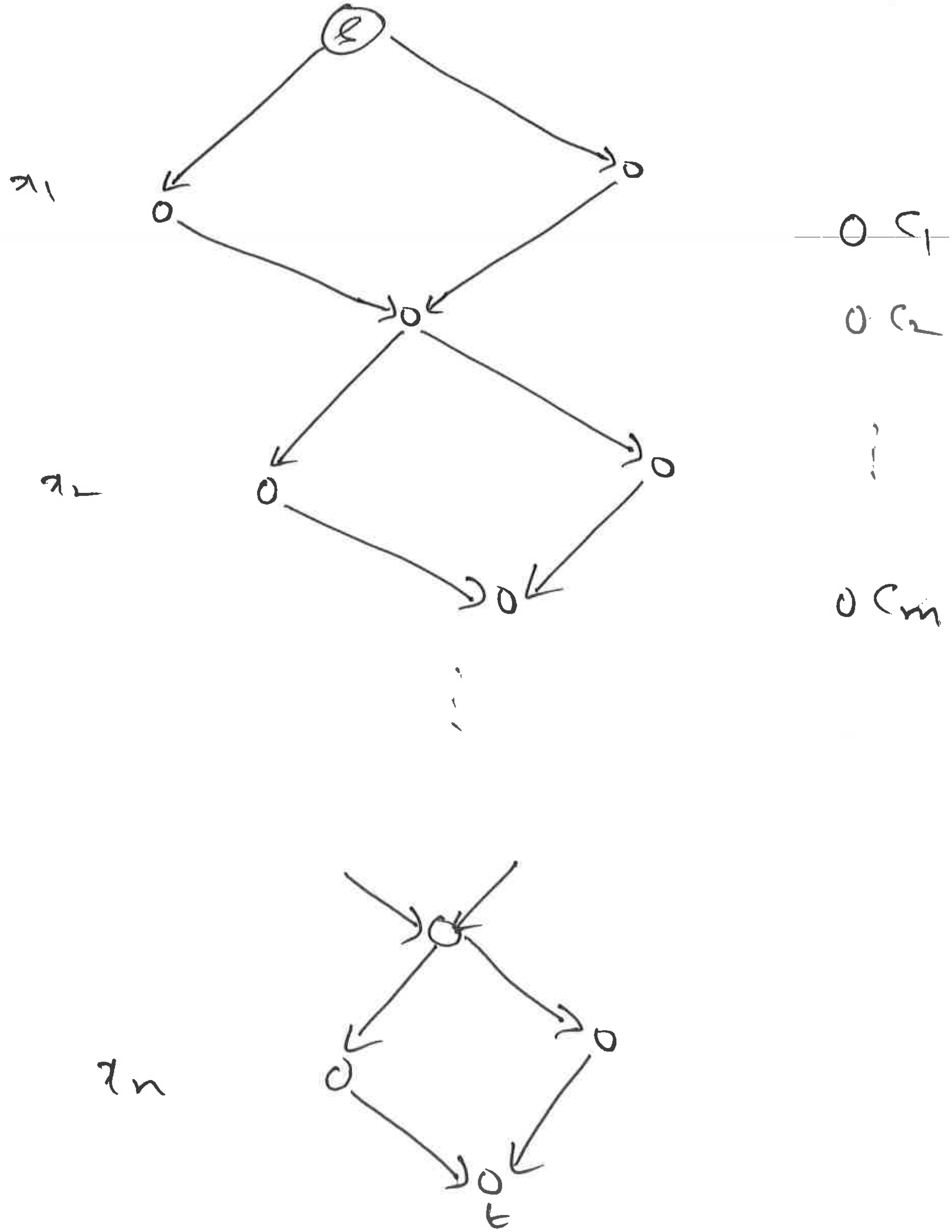
Connect the j^{th} clause pair in the i^{th} diamond to the j^{th} clause node as shown:



Case 2: Variable x_j appears as a −ve literal in C_j

Connect the j^{th} clause pair in the i^{th} diamond to the j^{th} clause node as shown:




s
x_1
c_1
c_2
x_2
c_m
x_n
t

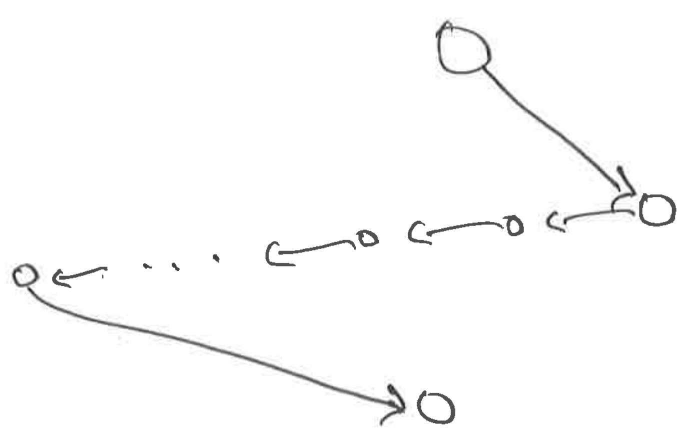
Correctness:

Suppose F is satisfiable. Let A be a satisfying truth assignment for $x_1, \ldots, x_n$.

- The path starts at s, the top node for the diamond of x_1.
- For the diamond x_i:

x_i = True in A: From the top node go to the left neighbor, traverse left to right (the "upper" path), from the right end go to the bottom node.

x_i = False in A: From the top node go to the right neighbor, traverse right to left (the "lower" path), from the left end go to the bottom node.

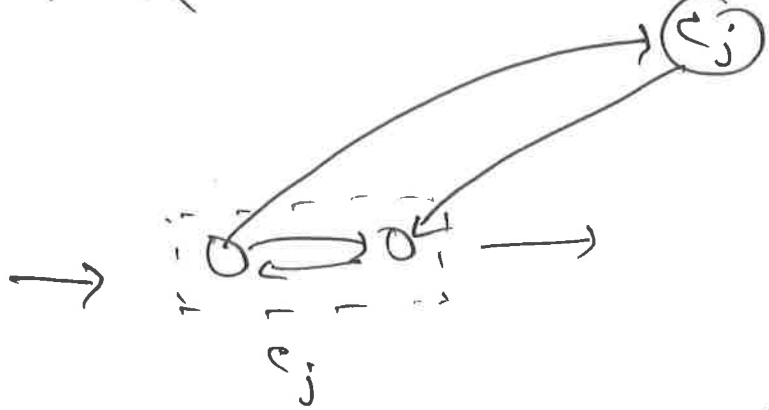


To Visit Clause nodes

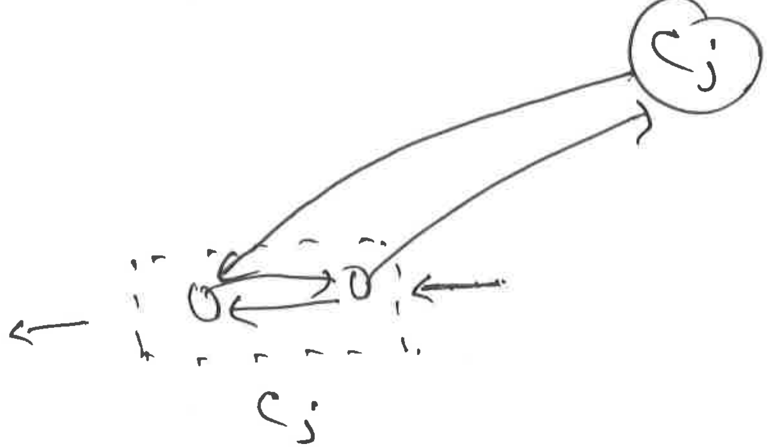
In each clause select one of the literals assigned true by $\mathcal{A}$.

In C_j if x_i is selected ~~= True~~ make a detour from the j^{th} pair of the i^{th} diamond to C_j and return to the ~~path~~ pairing node.

$x_i = \text{True}$ in C_j



$x_i = \text{False}$ in C_j



- A path in G is <u>normal</u> if: ~~it~~

goes thru' the diamonds in order from top to bottom except for detours to the clause nodes.

- If the DHP is normal then a natural satisfying assignment:

~~For x_i's diamond~~

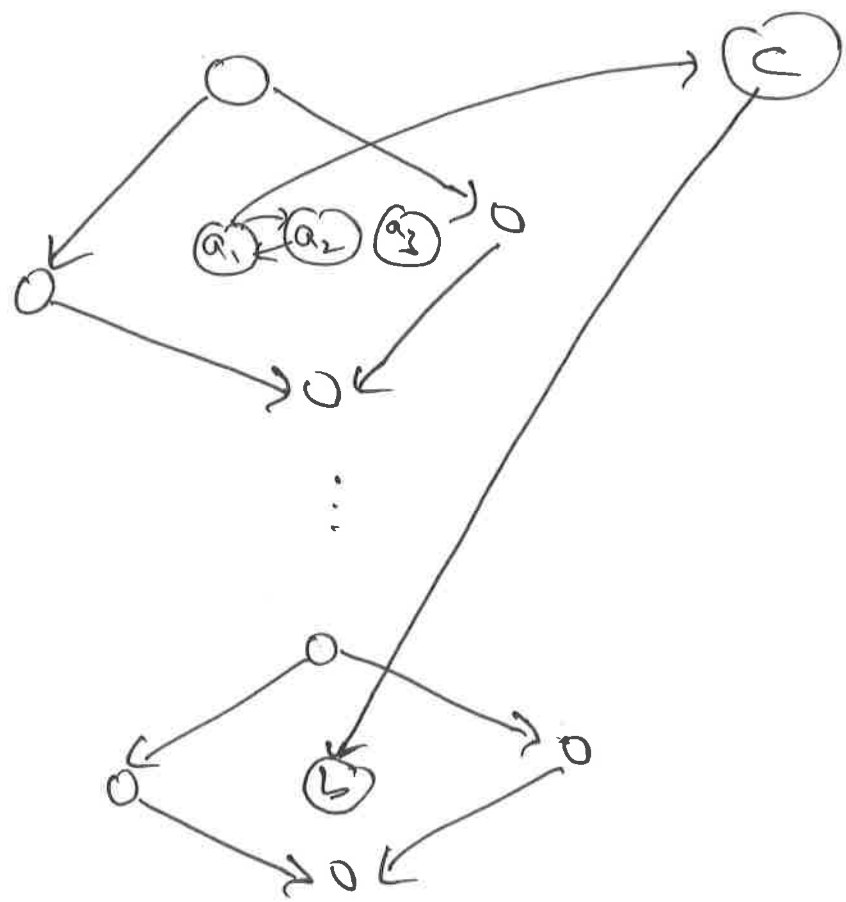
Set $x_i = \left\{\dfrac{\text{True}}{\text{False}}\right\}$ if the path is from $\left\{\dfrac{\text{left}}{\text{right}}\right\}$ to $\left\{\dfrac{\text{right}}{\text{left}}\right\}$.

- Any DHP must be normal.

Suppose not.

path goes from a node in the middle of the diamond to a clause node and from there to a node in another diamond.

Suppose the path is from left to right and it goes to clause node C from a_1. Instead of returning to a_2 from C suppose it goes to node b in another diamond. (See diagram.) a_3 is a separator node.



Path:

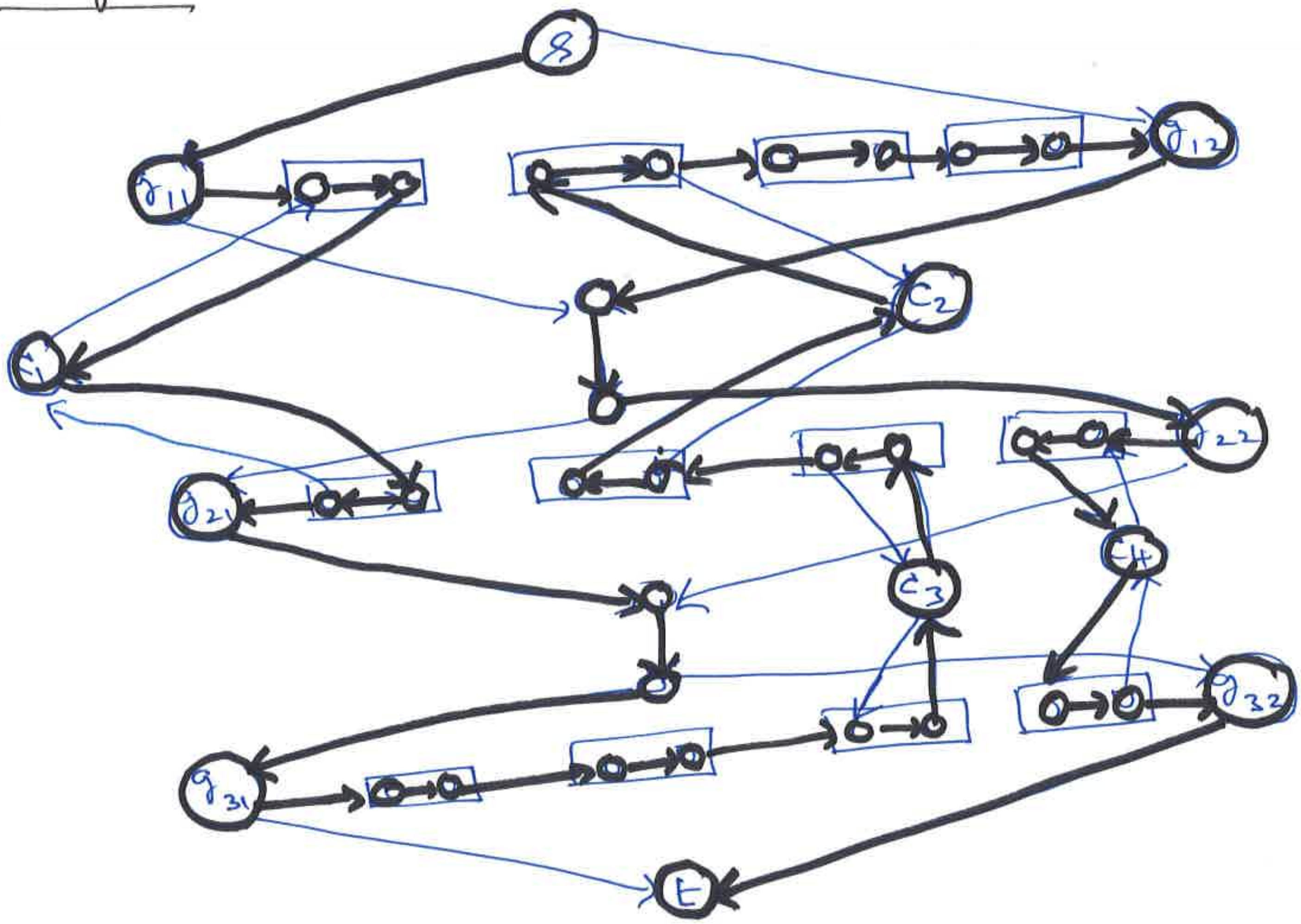
$\ell \ldots a_1 C b \ldots ? a_2 ? \ldots$

Path cannot visit a_2 ~~only edges~~ from C or a_1. Must enter a_2 from a_3. But cannot leave a_2 since the only edge out of a_2 is to a_3.

Other cases similar.

Without seperator nodes, a DHP need not be normal.

Example:



$(\bar{x}_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_2 \vee \bar{x}_3)$