Flow network: Directed $G = (V,E)$ with $s \in V$ & $t \in V$ and capacities $c_e > 0$.

A flow is an assignment $f_e$ for each $e \in E$ where:

1) for all $e \in E$, $0 \leq f_e \leq c_e$
2) for all $v \in V - \{s \cup t\}$, $f_{\text{in}}(v) = f_{\text{out}}(v)$

$$f_{\text{in}}(v) = \sum_{u \in V \setminus \{s\}} f_{uv} \quad \& \quad f_{\text{out}}(v) = \sum_{w \in V \setminus \{t\}} f_{vw}$$

$\text{Size}(f) = f_{\text{out}}(s) = f_{\text{in}}(t)$

Max-flow problem:
Given a flow network, find a flow $f$ with max $\text{size}(f)$.

Closely related is the following min st-cut problem.

What's a cut?

A cut is a partition of the vertices into 2 sets denoted as $L \cup R$ ($V = L \cup R \& L \cap R = \emptyset$)

This an st-cut if $s \in L \& t \in R$. 
For an st-cut define its capacity as:
\[
\text{capacity}(L,R) = \sum_{(u,v) \in E: \ v \in L, \ u \in R} c_{uv} = \text{capacity from } L \rightarrow R.
\]

**Min-st-cut problem:**
Given a flow network find an st-cut with min capacity.

**Max-flow min-cut theorem:**
\[
\text{size of } = \min \text{ capacity of an st-cut}
\]

Let's prove the theorem.

The easy direction is that \( \text{size of } \leq \min \text{ capacity of an st-cut} \).

We'll show that for any flow \( f \) & any st-cut \( (L,R) \),
\[
\text{size}(f) \leq \text{capacity}(L,R) \quad (*)
\]

and hence:
\[
\max_{f} \text{size}(f) \leq \min \text{ capacity}(L,R) \quad \text{(LR)}
\]

To intuitively see (*) note that \( f \) has to cross \( L \rightarrow R \) & \( \leq \text{capacity } (L,R) \) units can cross it.

Let's prove it formally.
For an s-t cut \((L, R)\) define

\[ f_{\text{in}}(L) = \sum_{u \in \text{U}} f_{uv} = \text{flow into } L \]

\[ f_{\text{out}}(L) = \sum_{v \in \text{V}} f_{vu} = \text{flow out of } L \]

We'll show that: \(\text{size}(f) = f_{\text{out}}(L) - f_{\text{in}}(L)\).

Why is this true?

\[ f_{\text{out}}(L) - f_{\text{in}}(L) = \sum_{v \in \text{V}} f_{vu} - \sum_{u \in \text{U}} f_{uv} \]

\[ = \sum_{v \in \text{V}} f_{vu} - \sum_{u \in \text{U}} f_{uv} + \sum_{v \in \text{V}} f_{vu} - \sum_{u \in \text{U}} f_{uv} \]

\[ = f_{\text{out}}(S) + \sum_{v \in \text{V}} \left( f_{\text{out}}(V) - f_{\text{in}}(V) \right) \]

These sums are equal just counting from different ends.

\[ = \text{size}(f) \]
Therefore, for any flow \( f \), any st-cut \((L,R)\),
\[
\text{size}(f) = f^\text{out}(L) - f^\text{in}(L) \leq f^\text{out}(L) \leq \text{capacity}(L,R).
\]
This proves \((*)\).

Now we'll show that:
\[
\max_f \text{size}(f) \geq \min_{L,R} \text{capacity}(L,R).
\]
Take the flow \( f^* \) produced by the Ford-Fulkerson alg.
This flow \( f^* \) has no "augmenting path." This means that there is no st-path in the residual network \( G^f \).
We'll construct an st-cut \((L,R)\) where:
\[
\text{size}(f^*) = \text{capacity}(L,R), \quad (**)
\]
hence,
\[
\max_f \text{size}(f) \geq \min_{L,R} \text{capacity}(L,R)
\]
which is the other half of the max-flow theorem.

Let's prove \((**):\)

For flow \( f^* \), let \( L \) be those vertices reachable from \( s \) in \( G^f \).
We know \( f^* \mid L \) since there is no st-path in \( G^f \).
Hence let \( R = V - L \) & \((L,R)\) is a st-cut.
Consider an edge from $L \rightarrow R$:
for $vw \in E$ where $v \in L \& w \in R$
we must have $f^*_vw = c_{vw}$
otherwise $vw$ in $G^f$ & then $w$ is reachable from $s$.

Consider an edge from $R \rightarrow L$:
for $zy \in E$ where $z \in R \& y \in L$:
we must have $f^*_zy = 0$
otherwise $zy \in G^f$ & then $z$ is reachable from $s$.

Thus, $f^*_{\text{out}}(L) = \text{capacity}(L,R)$ by
& $f^*_{\text{in}}(L) = 0$ by

Therefore, $\text{size}(f^*) = f^*_{\text{out}}(L) - f^*_{\text{in}}(L)$

$= \text{capacity}(L,R)$
which proves (**).

We also know that any flow $f^*$ with no augmenting paths is a max-flow, since it has
$\text{size}(f^*) = \min \text{st-cut capacity}$.
An application of max-flow:

**Image segmentation:**

Given an image separate it into objects.
Simpler problem: separate it into foreground & background.

Image is on a graph $G = (V, E)$ (think of $G$ as a grid)

$V =$ pixels
$E =$ neighboring pixels.

For $i \in V$

given likelihood $a_i$ that $i$ is in the foreground.
& $b_i$ that $i$ is in the background.

$a_i \geq 0$ & $b_i \geq 0$.

for $(i, j) \in E$

given separation penalty $P_{ij} \geq 0$. 
for partition \((A, B)\) where \(V = A \cup B\)

- \(A = \) foreground
- \(B = \) background

Define its weight as

\[
\omega(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i, j) \in A} p_{ij} - \sum_{i \in A, j \in B} p_{ij}.
\]

**Goal:** find partition \((A, B)\) with max weight \(\omega(A, B)\).

Reduce to min st-cut problem.

First convert from a maximization to a minimization problem.

Let \(Q = \sum_{i \in V} (a_i + b_i)\)

Note: \(\sum_{i \in A} a_i + \sum_{j \in B} b_j = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j\)
Thus,
\[ w(A,B) = Q - \sum_{i \in A} b_i - \sum_{j \in B} a_j - \sum_{(i,j) \in E} P_{ij} \]

Let \[ w'(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{(i,j) \in E} P_{ij} \]

\((A,B)\) which maximizes \(w(A,B)\)
is the same as \((A,B)\) which minimizes \(w'(A,B)\)

\[ w(A,B) = Q - w'(A,B) \]

\(\max_{(A,B)} w(A,B) = Q - \min_{(A,B)} w'(A,B)\)

To reduce to max-flow:

for edge \((i,j) \in E:\) add edges \(i \rightarrow j\) capacity \(P_{ij}\)

\(j \rightarrow i\) capacity \(P_{ij}\)

add source \(s\) for every \(i \in V\), add edges \(s \rightarrow i\)
capacity \(a_i\)

add sink \(t\), for every \(j \in V\), add edge \(j \rightarrow t\)
capacity \(b_j\)
In this flow network we constructed, for an st-cut \((A,B)\), what edges cross \(A \to B\)?

For \(j \in B\) get edge \(S \to j\) of capacity \(a_j\).

For \(i \in A\) get \(i \to t\) of capacity \(b_i\).

For \((i,j)\) where \(i \in A, j \in B\) get \(i \to j\) of capacity \(p_{ij}\).

And if \(i \in B, j \in A\) get \(j \to i\) also of capacity \(p_{ij}\).

Thus,

\[
\text{capacity } (A,B) = w'(A,B)
\]

So run max-flow, the size of the max-flow = capacity of min st-cut = \(\min_{(A,B)} w'(A,B)\).

Then, \(\max_{(A,B)} w(A,B) = Q - \min_{(A,B)} w'(A,B)\).