Given a graph $G=(V,E)$

let $\mathcal{M}(G)$ = all matchings of G (any size)

Sampling Problem: generate a matching from $\pi = \text{uniform}(\mathcal{M})$.

Counting Problem: FPRAS for $|\mathcal{M}|$ = # of matchings.

Harder problem (later): $\mathcal{P}$ = perfect matchings (for bipartite G)

Markov chain for sampling problem:

let $\Omega = \mathcal{M}$ = collection of all matchings of input graph G.

From $X_t \in \Omega$,

1. Choose an edge $e=(v,w)$ u.a.r. from E.
2. Set $X' = X_t \oplus e$, i.e., $X' = \begin{cases} X_t \cup e & \text{if } e \notin X_t \\ X_t \setminus e & \text{if } e \in X_t. \end{cases}$
3. If $X' \in \Omega$ then $X_{t+1} = X'$ w. prob. $1/2$

   otherwise set $X_{t+1} = X_t$.

This MC is ergodic, and symmetric.

Hence, $\pi = \text{uniform}(\Omega)$.

Later we'll show $T_{mix} = \text{Poly}(n)$ for all G.

Let's use this sampling algorithm to design an FPRAS for the counting problem.

Order the edges $E = \{e_1, e_2, \ldots, e_m\}$
(arbitrary order)

Let $G_0 = G$, and for $i > 0$,
let $G_i = G \setminus e_i$ (remove edge e_i)

Thus, G_m = empty graph
& thus $|\mathcal{M}(G_0)| = 1$.

Note,

$$|\mathcal{M}(G)| = \frac{|\mathcal{M}(G_0)|}{|\mathcal{M}(G_1)|} \times \frac{|\mathcal{M}(G_1)|}{|\mathcal{M}(G_2)|} \times \cdots \times \frac{|\mathcal{M}(G_{m-1})|}{|\mathcal{M}(G_m)|} \times \overset{1}{|\mathcal{M}(G_m)|}$$

Let $\alpha_i = \dfrac{|\mathcal{M}(G_i)|}{|\mathcal{M}(G_{i-1})|}$.

Then, $|\mathcal{M}(G)| = \dfrac{1}{\alpha_1 \alpha_2 \cdots \alpha_m}$

Note, $\mathcal{M}(G_i) \subseteq \mathcal{M}(G_{i-1})$ since $M \in \mathcal{M}(G_i)$ is also in $\mathcal{M}(G_{i-1})$.

& thus $\alpha_i \leq 1$.

Moreover, $\alpha_i \geq 1/2$ because:

$$|\mathcal{M}(G_{i-1}) \setminus \mathcal{M}(G_i)| \leq |\mathcal{M}(G_i)|$$

by mapping $f: \mathcal{M}(G_{i-1}) \setminus \mathcal{M}(G_i) \rightarrow \mathcal{M}(G_i)$ as $f(M) = M \setminus e_i$.

Therefore,

$$\frac{1}{2} \leq \alpha_i \leq 1.$$

To estimate $\alpha_i = \dfrac{|\mathcal{M}(G_i)|}{|\mathcal{M}(G_{i-1})|}$,

generate samples $M_1^i, \dots, M_\ell^i$ from

μ_i where $\|\mu_i - \pi_i\| \leq \delta_i$

for $\pi_i = \text{uniform}(\cancel{\Omega_{i-1}}, \mathcal{M}(G_{i-1}))$

& $\delta_i = \dfrac{\epsilon}{6m}$ ($\epsilon > 0$ is the desired accuracy of $|\mathcal{M}(G)|$)

Let $X_j^i = \begin{cases} 1 & \text{if } M_j^i \in \mathcal{M}(G_i) \\ 0 & \text{if not} \end{cases}$

Note, $\alpha_i - \delta_i \leq E[X_j^i] \leq \alpha_i + \delta_i$

& thus, $\alpha_i\left(1 - \frac{\epsilon}{3m}\right) \leq E[X_j^i] \leq \alpha_i\left(1 + \frac{\epsilon}{3m}\right)$

By Chebyshev's, for $\ell = \cancel{O([illegible])}\ O\left(\frac{m}{\epsilon^2}\right)$,

$N = \left(\overline{X}_1 \overline{X}_2 \cdots \overline{X}_m\right)^{-1}$ where $\overline{X}_i = \frac{1}{\ell}\sum_{j=1}^{\ell} X_j^i$

is an ~~FPRAS~~ $(1 \pm \epsilon)$-approx. for $|\mathcal{M}(G)|$. with prob. $\geq 3/4$

then use median/Chernoff to boost

How to prove rapid mixing ($T_{mix} = \mathrm{Poly}(n)$)
for the MC on matchings?

MC defined by (P, Ω, π)
Graph defined by
Vertices $= \Omega$
Edges $= \{M \to M' : P(M, M') > 0\}$

Conductance = normalized edge expansion.

For set S where $\pi(S) \leq 1/2$,

$$\Phi(S) = \Pr(X_{t+1} \notin S \mid X_t \in S, X_t \sim \pi)$$

$$= \frac{\sum_{M \in S, M' \notin S} \pi(M) P(M, M')}{\pi(S)}$$

For the MC on matchings, $P(M, M') = \frac{1}{m}$
& $\pi(M) = \frac{1}{|\Omega|}$

thus, $\Phi(S) = \frac{1}{m} \frac{\# E(S, \bar{S})}{|S|}$

Let $\Phi = \min_{S: \pi(S) \le 1/2} \Phi(S)$

<u>Theorem:</u>

$$\Omega\left(\frac{1}{\Phi}\right) = T_{mix} = O\left(\frac{1}{\Phi^2} \log\left(\frac{1}{\pi_{min}}\right)\right)$$

Easy inequality: Since $\pi(S) \le 1/2$, to get close to π have to atleast visit $\bar{S}$.

Set $X_0 \in S$, $X_0 \sim \pi$.

Then $\Phi(S)$ is the prob. of leaving in 1 step, & $\frac{1}{\Phi(S)}$ is the expected # of steps to leave S & visit $\bar{S}$.

To lower bound the mixing time, find a set S with bad conductance.

To upper bound the mixing time,
Prove that the conductance $\Phi(S) \geq \frac{1}{\text{Poly}(n)}$
for every $S \subset \Omega$.
Doesn't give as tight bounds as coupling.

Canonical paths:

For every pair $I, F \in \Omega$,
define a path γ_{IF} along edges in (Ω, P).
assume $P(M, M') = \frac{1}{m}$ $\forall (M, M') \in P$,
& $\pi = \text{Uniform}(\Omega)$.

For edge $T = M \to M'$,
define its congestion:

$$cp(T) = \{(I, F) : T \in \gamma_{IF}\} = \text{set of paths that go through } T.$$

let $\rho = \max_T \frac{|cp(T)|}{|\Omega|}$

Lemma: $\Phi \geq \frac{1}{2m\rho}$

Proof: Fix $S \subset \Omega$ where $\pi(S) \leq 1/2$

& thus $|S| \leq |\overline{S}|$, and $|\overline{S}| \geq \frac{|\Omega|}{2}$.

Let's bound $|E(S,\overline{S})|$:

There are $|S| \times |\overline{S}|$ pairs (I,F) with $I \in S$, and $F \in \overline{S}$

each of these crosses $S \to \overline{S}$

at least once on γ_{IF}

every edge $T = S \to \overline{S} \in E(S,\overline{S})$

has $\leq \rho|\Omega|$ through it

Therefore, $\geq \frac{|S||\overline{S}|}{\rho|\Omega|} \geq \frac{|S|}{2\rho}$

transitions from $S \to \overline{S}$.

So $|E(S,\overline{S})| \geq \frac{|S|}{2\rho}$.

Random walk on hypercube:

$$\Omega = \{0,1\}^n$$

From $X_t \in \Omega$,

1. Choose $i \in_R \{1,\dots,n\}$ & $b \in_R \{0,1\}$
2. for all $j \neq i$, $X_{t+1}(j) = X_t(j)$
3. Set $X_{t+1}(i) = b$.

For $I, F \in \Omega$, canonical path γ_{IF}:

- for $i = 1 \to n$:

  change $I(i) \to F(i)$

Consider transition $T = X \to X'$ which flips i^{th} bit.

~~Set $E = (F(1), \dots, F(i), I(i+1), \dots, I(n))$~~

Set $E = (I(1), \dots, I(i), F(i+1), \dots, F(n))$

<u>Claim</u>: $E: cp(T) \to \Omega$ & E is injective (can invert)

where $cp(T) = \{(I,F) : \gamma_{IF} \ni T\}$.

Proof of claim:

Note, transition T agrees with F on 1^{st} $i-1$ bits, and with I on last bits $i+1, \dots, n$.

Thus, from E & T can infer F on all bits & I on all bits.

(can use $X \to X'$ transition to get $I(i)$ & $F(i)$)

Therefore, E is injective & clearly $E \in \Omega$. $\blacksquare$

Thus, $|cp(T)| \leq |\Omega|$, and so $\rho = O(1)$.

and this implies $\Phi \geq \cancel{\frac{1}{2m}}\ \Omega\left(\frac{1}{n}\right)$.

Finally, $T_{mix} = \cancel{O(m^2 n)}$ $O(n^3)$ since $\pi_{min} = 2^{-n}$. ($\Delta m = n$ in this problem)

Note, using coupling we got an $O(n \log n)$ bound on the mixing time.