Given a graph $G = (V,E)$

let $\mathcal{M}(G)$ = all matchings of $G$ (any size)

Sampling problem: generate a matching from $\pi = \text{uniform}(\mathcal{M})$.

Counting problem: FPRAS for $|\mathcal{M}| = \# \text{of matchings}$. 

Harder problem (later): $\Phi = \text{perfect matchings (for bipartite)}$.

Markov chain for sampling problem:

let $\mathcal{M} = \text{collection of all matchings of input graph } G$.

From $X_t \in \mathcal{M}$,

1. Choose an edge $e = (x, w)$ uniformly from $E$.

2. Set $X' = X_t \oplus e$, i.e., $X' = \begin{cases} X_{t+1} \oplus e & \text{if } e \notin X_t \\ X_t \oplus e & \text{if } e \in X_t. \end{cases}$

3. If $X' \in \mathcal{M}$ then $X_{t+1} = X'$ w/ prob. $\frac{1}{2}$
   otherwise set $X_{t+1} = X_t$. 

This MC is ergodic and symmetric. Hence, \( \Pi = \text{uniform}(G) \).

Later we'll show \( \text{Tr} x = \text{poly}(n) \) for all \( G \).

Let's use this sampling algorithm to design an FPRAS for the counting problem.

Order the edges \( E = \{ e_1, e_2, \ldots, e_m \} \).

(arbitrary order)

Let \( G_0 = G \), and for \( i > 0 \):

Let \( G_i = G \backslash e_i \) (remove edge \( e_i \)).

Thus, \( G_m = \text{empty graph} \)

& thus \( |M(G_0)| = 1 \).

Note,

\[
|M(G)| = \frac{|M(G_0)|}{|M(G_1)|} \times \frac{|M(G)|}{|M(G_2)|} \times \cdots \times \frac{|M(G_m)|}{|M(G_m)|}
\]
Let \( \alpha_i = \frac{|M(G_i)|}{|M(G_{i-1})|} \).

Then, \( |M(G)| = \frac{1}{\alpha_1 \alpha_2 \ldots \alpha_m} \).

Note, \( M(G_i) \subseteq M(G_{i-1}) \) since \( M \in M(G_{i-1}) \) is also in \( M(G_{i-1}) \).

And thus \( \alpha_i \leq 1 \).

Moreover, \( \alpha_i \geq \frac{1}{2} \) because:

\[ |M(G_{i-1}) \setminus M(G_i)| \leq |M(G_i)| \]

by mapping \( f : M(G_{i-1}) \setminus M(G_i) \rightarrow M(G_i) \)

as \( f(M) = M \setminus e_i \).

Therefore,

\[ \frac{1}{2} \leq \alpha_i \leq 1. \]
To estimate $\alpha_i = \frac{|\mathbf{M}(G)|}{|\mathbf{M}(G_i-1)|}$,

Generate samples $M_1^i, \ldots, M_m^i$ from $M_i$ where \( \|M_i - \mathbf{π}_i\| \leq \delta_i \)

for $\mathbf{π}_i = \text{uniform}(\mathbb{E} \rightarrow \mathbf{M}(G_{i-1}))$

and $\delta_i = \frac{e}{6m}$ ($\varepsilon > 0$ is the desired accuracy of $|\mathbf{M}(G)|$)

Let $X_j = 1$ if $M_j^i \in \mathbf{M}(G_i)$

and $X_j = 0$ if not

Note, $\alpha_i - \delta_i \leq E[X_j] \leq \alpha_i + \delta_i$

& thus, $\alpha_i(1 - \frac{e}{3m}) \leq E[X_j] \leq \alpha_i(1 + \frac{e}{3m})$

By Chebyshev's, for $\lambda = O\left(\frac{m}{e^2}\right)$

\[
N = \left(\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_m\right)^{-1}
\]

where $\bar{X}_i = \frac{1}{N} \sum_{j=1}^{N} X_j$

is an \textit{EPRAS} for $|\mathbf{M}(G)|$, with prob. $\geq \frac{3}{4}$

$(1 \pm \varepsilon)$-approx. then use median/Chernoff to boost.
How to prove rapid mixing \((T_{\text{mix}} = \text{Poly}(n))\) for the MC on matchings?

MC defined by \((P, \mathcal{S}, \pi)\)

Graph defined by

Vertices = \(\mathcal{S}\)

Edges = \(\{m \Rightarrow m': P(m, m') > 0\}\)

Conductance = normalized edge expansion.

For set \(S\) where \(\pi(S) \leq \frac{1}{2}\),

\[
\Phi(S) = \Pr(X_{t+1} \notin S | X_t \in S, X_t \sim \pi)
\]

\[
= \sum_{m \in S, m' \in \mathcal{S}} \pi(m)P(m, m') \frac{\pi(S)}{\pi(S)}
\]

For the MC on matchings, \(P(m, m') = \frac{1}{m} \quad \# \pi(m) = \frac{1}{\sqrt{2}}\)

Thus, \(\Phi(S) = \frac{1}{m} \frac{\# \mathcal{E}(SS)}{|S|}\)
Let \( \Phi = \min_{S: \Pi(S) \leq \frac{1}{2}} \Pi(S) \)

**Theorem:**

\[ T_{\text{mix}} = O\left( \frac{1}{\Phi^2} \log \left( \frac{1}{\Pi_{\text{min}}} \right) \right) \]

Easy inequality: Since \( \Pi(S) \leq \frac{1}{2} \), to get close to \( \Pi \) have to at least visit \( S \).

Set \( \{x_0, \ldots, x_n\} \sim \Pi \).

Then \( \Phi(S) \) is the prob. of leaving in 1 step, and \( \frac{1}{\Phi(S)} \) is the expected \# of steps to leave \( S \) & visit \( S \).

To lower bound the mixing time, find a set \( S \) with bad conductance.
To upper bound the mixing time,
prove that the conductance \( \Phi(S) \geq \frac{1}{\text{poly}(n)} \)
for every \( S \subseteq Z \).

Doesn't give as tight bounds as coupling.

**Canonical paths:**

For every pair \( I, F \in Z \),
define a path \( \gamma_{IF} \) along edges in \((Z, P)\),
assume \( P(M, M') = \frac{1}{m} \) \( \forall (M, M') \in P \),
\& \( \gamma = \text{uniform}(Z) \).

For edge \( T = M \rightarrow M' \),
define its congestion:
\[
\text{cp}(T) = \sum (I, F) : T \in \gamma_{IF} \text{ set of paths that go through } T.
\]

\[
\text{let } p = \max_{T} \frac{\text{cp}(T)}{|S|}.
\]
Lemma: \( \Phi \geq \frac{1}{2m} \)

Proof: Fix SCCL where \( \Pi(S) \leq \frac{1}{2} \) and thus \( |S| \leq |S| \) and \( |S| \geq 12/2 \).

Let's bound \( |E(S,S)| \):

There are \( |S| \times |S| \) pairs \((I,F)\) with \( \frac{|E_S|}{S_S} \) each of these crosses \( S \rightarrow S \) at least once on \( \Phi_{IF} \).

Every edge \( \tau = S \rightarrow S \in E(S,S) \) has \( \leq \frac{1}{|S|^2} \) through it.

Therefore, \( \geq \frac{|S| |S|}{|S|^2} \geq \frac{|S|}{2p} \) transitions from \( S \rightarrow S \).

So \( |E(S,S)| \geq \frac{|S|}{2p} \).
Random walk on hypercube:

\[ S = \{0, 1\}^n \]

From \( X_0 \in S \):
1. Choose \( i \in \{1, ..., n\} \) & \( b \in \{0, 1\} \)
2. For all \( j \neq i \), \( X_{+1}(j) = X_+(j) \)
3. Set \( X_{+1}(i) = b \)

For \( I, F \in S \), canonical path \( X_{IF} \):

- For \( i = 1 \rightarrow n \):
  - change \( I(i) \rightarrow F(i) \)

Consider transition \( T = X \rightarrow X' \) which flips \( i^{th} \) bit:

Set \( E = (F(1), ..., F(i), I(i+1), ..., I(n)) \)

Set \( E = (I(1), ..., I(i), F(i+1), ..., F(n)) \)

Claim: \( E: cp(T) \rightarrow S \) & \( E \) is injective (can invert)

where \( cp(T) = \{ (I,F) : Y_{IF} \in T \} \)
Proof of claim:

Note, transition T agrees with F on first i-1 bits, and with I on last bits i,...,n.

Thus, from E & T can infer F on all bits & I on all bits.

(can use \( X \to X' \) transition to get \( I(i) \) & \( F(i) \))

Therefore, E is injective & clearly \( E \in \mathbb{R} \).

Thus, \( |c_F(T)| \leq 1521 \), and so \( p = O(1) \).

And this implies \( \Theta \geq \frac{1}{2} + \Omega(\frac{1}{n}) \).

Finally, \( T_{\text{mix}} = O\left( \frac{m^2}{n} \right) \) since \( T_{\text{min}} = 2^{-n} \).

Note, using coupling we get an \( O(n \log n) \) bound on the mixing time.