Given a graph $G = (V, E)$ of maximum degree $\Delta$ and integer $k > 0$, generate a random $k$-coloring of $G$.

**Goal:** Poly-time algorithm when $k > \Delta$.

**Markov Chain:** Glauber Dynamics = single-vertex update

1. Let $\mathcal{Z}$ = collection of proper vertex $k$-colorings of $G$.
2. Set $X_0$ = some proper vertex $k$-coloring.
3. From $X_t \in \mathcal{Z}$,
   
   1. Choose a vertex $v$ uniformly at random (var) from $V$
   2. Choose a color $c$ (var) from $\{1, \ldots, k\}$
   3. For all $w \neq v$ set $X_{t+1}(w) = X_t(w)$.
   4. If no neighbors of $v$ have color $c$
      then $X_{t+1}(v) = c$
      else $X_{t+1}(v) = X_t(v)$.

**Claim:** When $k \geq \Delta + 2$ the MC is ergodic.

Since $P$ is symmetric then $\pi = \text{Uniform}(\mathcal{Z})$ when $k \geq \Delta + 2$. 
Analyze mixing time = \# of steps from worst \(X_0\) to get "close" to \(\pi\).

Measure distance using total variation distance:

\[
\text{for a pair of distributions } \mu \& \nu \text{ on } \mathbb{Z},
\]
\[
\text{\(Q_{TV}(\mu, \nu) = \frac{1}{2} \sum_{\sigma \in \mathbb{Z}} |\mu(\sigma) - \nu(\sigma)|\)}
\]
\[
= \sum_{\sigma : \mu(\sigma) \geq \nu(\sigma)} \mu(\sigma) - \nu(\sigma)
\]
\[
= \max_{S \subseteq \mathbb{Z}} \mu(S) - \nu(S).
\]

for \(\epsilon > 0\), \(X_0 \in \mathbb{Z}^d\),
\[
T_{mix}(\epsilon) = \min \{ t \in \mathbb{Z}^+ : Q_{TV}(P_{1}^t(X_0), \pi) \leq \epsilon \}
\]

Distribution of \(X_+\) given \(X_0\).

\[
T_{mix}(\epsilon) = \max_{X_0 \in \mathbb{Z}} T_{mix}(\epsilon).
\]

Let \(T_{mix} = T_{mix}(\frac{1}{4})\)

then \(T_{mix}(\epsilon) \leq T_{mix} \times \log \left(\frac{1}{\epsilon}\right)\). (Prove using coupling)
Coupling: a method to bound distance b/w distributions.

For \( \mu \) & \( \nu \) on \( \Sigma \),

let \( \omega \) be a distribution on \( \Sigma \times \Sigma \).

Then \( \omega \) is a coupling of \( \mu \) & \( \nu \) if:

for all \( \sigma \in \Sigma \), \( \sum_{\tau \in \Sigma} \omega(\sigma, \tau) = \mu(\sigma) \)

& for all \( \tau \in \Sigma \), \( \sum_{\sigma \in \Sigma} \omega(\sigma, \tau) = \nu(\tau) \)

(in words, for \( \omega \), the marginal in the 1st coordinate is \( \mu \)

& in the 2nd coordinate is \( \nu \))

Choose \( (X, Y) \sim \omega \)

Then, \( D_{TV}(\mu, \nu) \leq \Pr(X = Y) \)

Proof: \( \Pr(X = Y) = \sum_{\sigma \in \Sigma} \omega(\sigma, \sigma) \leq \sum_{\sigma \in \Sigma} \min \{ \mu(\sigma), \nu(\sigma) \} \)

& hence, \( \Pr(X \neq Y) \geq 1 - \sum_{\sigma \in \Sigma} \min \{ \mu(\sigma), \nu(\sigma) \} \)

\[ = \sum_{\sigma \in \Sigma} \mu(\sigma) - \min \{ \mu(\sigma), \nu(\sigma) \} \]

\[ = \sum_{\sigma : \mu(\sigma) \geq \nu(\sigma)} \mu(\sigma) - \nu(\sigma) = D_{TV}(\mu, \nu). \]
Note, a coupling \( w \) of \( \mu \& \nu \) s.t.
\[ d_{TV}(\mu, \nu) = \Pr(X \neq Y). \]

Exercise: Prove this fact by construction.

Now consider a MC defined by \( P \) on \( S_2 \).
Make 2 copies \((X^+, Y^+)\), with arbitrary \( X_0, Y_0 \).
From \((X^+, Y^+)\), define \((X^{+1}, Y^{+1})\) so that:
\[ X^+ \rightarrow X^{+1} \quad \text{is distributed according to} \ P \]
\[ Y^+ \rightarrow Y^{+1} \quad \text{but they can be correlated.} \]

Then,
\[ d_{TV}(P^+(X_0, \cdot), P^+(Y_0, \cdot)) \leq \Pr(X^+ \neq Y^+) \]
& if for all \( X_0, Y_0 \),
\[ \Pr(X^+ \neq Y^+) \leq \frac{1}{4} \]
then by setting \( Y_0 \sim \Pi \) we have:
\[ T_{mix} \leq t. \]
Toy example: Random walk on the hypercube. 

$S^2 = \{0,1\}^n = n$-bit vectors.

From $X_t \in \{0,1\}^n$:

1. Choose $i$ var from $\{1,2,\ldots,n\}$ 
   & $b$ var from $\{0,1\}$.

2. for all $j \neq i$, set $X_{t+1}(j) = X_t(j)$.

3. Set $X_{t+1}(i) = b$.

Lemma: $T_{\text{mix}} = O(n \log n)$

Proof: For a pair $X_t, Y_t \in \{0,1\}^n$

define a coupling $(X_t, Y_t) \rightarrow (X_{t+1}, Y_{t+1})$ by:

Using the same random $i$ & $b$.

Let $H_t = |\{i : X_t(i) \neq Y_t(i)\}|$

$E[H_{t+1}] \leq H_t - \frac{H_t}{n} = H_t(1 - \frac{1}{n})$

$E[H_0] \leq H_0 (1 - \frac{1}{n})^t \leq ne^{-\frac{t}{2n}}$ for $t = \frac{1}{2n}$.
Back to colorings:

For \( k \)-colorings \( X_t, Y_t \in \mathcal{Z} \),

Choose same \( v \) & \( c \) for attempted update

Let \( H_t = \left| \{ w \in V : X_t(w) \neq Y_t(w) \} \right| \)

= \# of vertices that \( X_t \& Y_t \) differ on.

Let \( D_t = \{ w \in V : X_t(w) \neq Y_t(w) \} \)

& thus \( H_t = |D_t| \)

& \( A_t = V \setminus D_t = \{ w \in V : X_t(w) = Y_t(w) \} \)

For \( w \in V \), Let \( a_t(w) = |A_t \cap NN(w)| \& \( d_t(w) = |D_t \cap NN(w)| \)

\[
\Pr(v \in A_{t+1} | v \in D_t) \geq \frac{k - 2\Delta + a_t(v)}{nk}
\]

\[
\Pr(v \in D_{t+1} | v \in A_t) \leq \frac{2d_t(v)}{nk}
\]

\[
E[H_{t+1} | X_t, Y_t] \leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} - \sum_{v \in D_t} \frac{k - 2\Delta + a_t(v)}{nk}
\]

(for \( k \geq 3\Delta_t \))

\[
\leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} + \sum_{v \in D_t} \frac{-1 - 2a_t(v)}{nk}
\]

\[
\leq H_t + \sum_{v \in A_t} \frac{2d_t(v)}{nk} + \sum_{v \in D_t} \frac{-1 - 2a_t(v)}{nk}
\]
So we have:

\[ E[H_{t+1} | X_t, Y_t] \leq H_t + \frac{1}{nk} \left[ \sum_{v \in A_t} 2 \delta_t(v) + \sum_{v \in D_t} 1 - 2a_t(v) \right] \]

Note, \[ \sum_{v \in A_t} \delta_t(v) = \sum_{v \in D_t} a_t(v) \]

hence,

\[ E[H_{t+1} | X_t, Y_t] \leq H_t - \frac{|D_t|}{nk} = H_t \left( 1 - \frac{1}{nk} \right) \]

Thus, \[ \Pr(X_t = Y_t) \leq E[H_t] \leq H_0 \left( 1 - \frac{1}{nk} \right) \]

\[ \leq ne^{-t/nk} \]

\[ \leq \frac{1}{4} \text{ for } t = nk \log(4n) \]

when \( k \geq 3\Delta + 1 \).

This proves \( T_{mix} = O(nk \log n) \)

when \( k > 3\Delta \).
How to improve?

Couplings compose:

Consider distributions \( M, \nu, \omega \) on \( \mathbb{Z} \),

& coupling \( \alpha \) of \( M \) & \( \nu \)

coupling \( \beta \) of \( \nu \) & \( \omega \)

then \( X = \alpha \circ \beta \) is a coupling of \( M \) & \( \omega \).

Choose \( \sigma \) from \( M \)
then apply \( \alpha \) to choose \( \xi \) from \( \nu \)
then apply \( \beta \) to choose \( \eta \) from \( \omega \).

Note,
\[
\gamma'(\xi, \eta) = \sum_{\xi} \alpha(\xi, \omega) \beta(\eta, \nu) 
\]
Define a coupling for all pairs of colorings $X^+, Y^+$

where $H(X^+, Y^+)=H_+=1$,

so they only differ on 1 vertex.

For arbitrary $W^+, Z^+$

define a sequence of colorings $X^0, X^1, \ldots, X^l$

where $X^0 = W^+, X^l = Z^+$

& $H(X^i, X^{i+1}) = 1$ \forall i

Note, $l = H(W^+, Z^+)$

Define a coupling for all pairs $X^+, Y^+$

where $H(X^+, Y^+)=1$

and the coupling satisfies:

$\mathbb{E}\left[ H(X_{n+1}^+, Y_{n+1}^+) \right] \leq 1 - \frac{1}{n^k}$
Then, for arbitrary $W_t, Z_t$

Consider the path $X_t, \ldots, X^0$

by composing couplings we have a

coupling $(W_t, Z_t) \rightarrow (W_{t+1}, Z_{t+1})$

\[ E[H(W_{t+1}, Z_{t+1})] \leq E\left[ \sum_{i=0}^{l-1} H(X^i_{t+i}, X^i_{t+i+1}) \right] \]

\[ \leq \sum_i E[H(X_{t+i}, X_{t+i+1})] \]

\[ \leq \sum_i (1 - \frac{1}{n^k}) \]

\[ = H_+(1 - \frac{1}{n^k}). \]

So it suffices to analyze pairs that differ on 1 vertex.
Couple \( B \) in \( \mathbf{X}^+(v) \) with \( R \) in \( \mathbf{Y}^+(v) \) & vice-versa for \( v \in \mathbb{N}(v^*) \).

Couple \( B \) in \( \mathbf{X}^+ \) with \( R \) in \( \mathbf{Y}^+ \) & \( R \) in \( \mathbf{X}^+ \) with \( B \) in \( \mathbf{Y}^+ \) & everything else, \( (\forall c) \) is same for \( \mathbf{X}^+, \mathbf{Y}^+ \).

Note,

\[
E[\mathcal{H}(x_{t+1}, y_{t+1})] = 1 - \left(1 - \frac{k-d}{nk}\right) + \frac{\Delta}{nk} \\
= 1 - \left(1 - \frac{k-2\Delta}{nk}\right) \leq 1 - \frac{1}{nk} \quad \text{for } k > 2\Delta.
\]