Undirected $G=(V,E)$. Notation: $n=|V|$, $m=|E|$.

Cut $(S,\bar{S})$:

for $S \subset V$, $\delta(S) = E(S,\bar{S})$

$= \{(v,w) \in E : v \in S, w \in \bar{S}\}$

<u>Goal</u>: Find a cut $(S,\bar{S})$ which minimizes $|\delta(S)|$

<u>Min-cut</u> problem

Closely related min st-cut problem:

Given ~~$s \in S, t \in \bar{S}$~~, find cut $(S,\bar{S})$
$s,t \in V$, where $s \in S, t \in \bar{S}$
which minimizes $|\delta(S)|$.

Can solve using max st-flow, takes $O(nm\log(n^2/m))$ time.

Run $n-1$ min st-cut problems (~~all~~ fix t, all others)
to get min-cut. Can do in total time.

Today: Karger's algorithm ('93)

- Simpler
- faster (Karger & Stein '96)

Basic operation: <u>Contract edges</u>

Multigraph: possibly multiple edges b/w vertices

Self-loop: edge (v,v) for $v \in V$.

For a multigraph $G=(V,E)$ without self-loops:

for edge $(v,w) \in E$:

Define G/e = G with contraction of $e=(v,w)$:

1. Replace v & w by new vertex z.
2. Replace all edges (v,y) or (w,y) by (z,y).
3. Remove self-loops to z.

<u>Observation:</u> Let $G' = G/e$ for $e=(v,w) \in E$.

For cut $(S,\bar{S})$ in G where $v,w \in S$ (or $v,w \in \bar{S}$)

then $(S,\bar{S})$ in G'

& vice-versa.

$\delta_G(S) = \delta_{G/e}(S)$ for all $S \subset V$ s.t. $v,w \in S$.

(Preserves cuts that don't span e).

Idea: Contract until 2 vertices remain.

The last 2 vertices v & w correspond to a cut $(S, \overline{S})$ in the original graph G.

Which edges to contract?

Fix a cut $(S, \overline{S})$ in G.

Don't want to choose edges $e \in \delta(S)$ to contract.

If $|\delta(S)|$ is min size, then it's small

So choose a random edge from E.

Karger's min-cut algorithm:

Let $G = (V, E)$ be the input graph.

Repeat until $|V| = 2$:

1. Choose an edge $e = (v, w)$ uniformly at random from E.

2. Set $G = G/e$.

Output the cut corresponding to the final 2 vertices.

Lemma: For graph $G=(V,E)$, let $\delta(S)$ be a cut of min-size.

$$\Pr(\text{Karger's alg. outputs } \delta(S)) \geq \frac{1}{\binom{n}{2}}$$

Proof: For Karger's alg., let $e_1, e_2, e_3, \dots, e_{n-3}$ denote the contracted edges in order.

$$\Pr(\text{alg. outputs } \delta(S)) = \Pr(e_1, e_2, \dots, e_{n-3} \notin \delta(S))$$

Let $k = |\delta(S)|$.

$$\Pr(e_1 \notin \delta(S)) = 1 - \Pr(e_1 \in \delta(S))$$
$$= 1 - \frac{k}{m}$$

What's m? Want a lower bound in terms of n.

For any $v \in V$, $(v, V-\{v\})$ is a cut.

This has size $\geq k$ so $\deg(v) \geq k$.

Hence, $m \geq kn/2$

& $\Pr(e_1 \notin \delta(S)) \geq 1 - \frac{k}{nk/2} = 1 - \frac{2}{n}$.

If $e_1 \notin \delta(S)$, then in $G_1 = G \setminus e_1$, $|\delta_{G_1}(S)| = k$.

How many edges in G_1?

$n-1$ vertices.

Every cut in G_1 corresponds to a cut in G

(but only some in $G \Rightarrow$ in G_1).

So min-cut in G_1 is $\geq k$

$$\Pr(e_2 \notin \delta(S) \mid e_1 \notin \delta(S)) \geq 1 - \frac{2k}{(n-1)k} = 1 - \frac{2}{n-1}$$

~~Pr~~ for events $\mathcal{E}$ & $\mathcal{F}$,

$$\Pr(\mathcal{E} \cap \mathcal{F}) = \Pr(\mathcal{E}) \Pr(\mathcal{F} \mid \mathcal{E})$$

$$= \Pr(\mathcal{E}) \frac{\Pr(\mathcal{F} \cap \mathcal{E})}{\Pr(\mathcal{E})}$$

$$\Pr(\text{alg. outputs } \delta(S)) = \Pr(e_1, \ldots, e_{n-3} \notin \delta(S))$$

$$= \Pr(e_1 \notin \delta(S)) \Pr(e_2 \notin \delta(S) \mid e_1 \notin \delta(S))$$

$$\times \cdots \times \Pr(e_{n-3} \notin \delta(S) \mid e_1, \ldots, e_{n-4} \notin \delta(S))$$

$$= \Pr(e_1 \notin \delta(S)) \prod_{j=1}^{n-4} \Pr(e_{j+1} \notin \delta(S) \mid e_1, \ldots, e_j \notin \delta(S))$$

$$= \prod_{j=0}^{n-3} \left(1 - \frac{2k}{n-j}\right)$$

$$= \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \times \cdots \times \left(1 - \frac{2}{3}\right)$$

$$= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right) \times \cdots \times \left(\frac{1}{3}\right)$$

$$= \frac{1}{n(n-1)}$$

$$= \frac{1}{\binom{n}{2}}$$

How do we boost the success probability?

Run $\ell\binom{n}{2}$ times & output the best (min) cut that we find.

$$\Pr\left(\text{all } \ell\binom{n}{2} \text{ runs don't find min cut } (S,\bar{S})\right)$$

$$\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^{\ell\binom{n}{2}} \leq e^{-\ell}$$

$$\Pr\left(\cancel{\text{atg}} \geq 1 \text{ run finds min-cut}\right) \geq 1 - e^{-\ell}$$

<u>Running time:</u> Each run takes $O(n^2)$ time

Why? Put in adj. matrix representation & then $O(n)$ time per contraction.

$\Rightarrow O(n^4 \log n)$ time to get error prob. $\leq 1/\text{poly}(n)$.

Faster version: [Karger-Stein]

Initial contractions are likely correct, but later ones are less likely, so need more runs at end.

$$\Pr(\delta(S) \text{ survives to } l+1 \text{ vertices})$$

$$\geq \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right) \times \cdots \times \left(\frac{l+1}{l+3}\right)\left(\frac{l}{l+2}\right)$$

$$= \frac{(l+1)(l)}{(n)(n-1)}$$

$$= \frac{\binom{l}{2}}{\binom{n}{2}}$$

when $l = \frac{n}{\sqrt{2}}$ then prob. of success $\geq \frac{\binom{n/\sqrt{2}}{2}}{\binom{n}{2}}$

$$= \frac{\left(\frac{n}{\sqrt{2}}\right)\left(\frac{n}{\sqrt{2}}-1\right)}{(n)(n-1)} = 1/2$$

So "double" when get down to $\frac{n}{\sqrt{2}}$ vertices.

Better algorithm:

From a multigraph G,
if G has ≥ 6 vertices:
Do twice & take best of 2:
1. Run Karger's down to $\frac{n}{\sqrt{2}}+1$ vertices
2. Recurse on this remaining graph.

Running time:

$$T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2) = O(n^2 \log n)$$

What's the success prob.?
We succeed down to $\frac{n}{\sqrt{2}}$ vertices with prob. $\geq 1/2$

Let $P(n)$ = Prob. of success with n vertices.

One run succeeds with prob. $\geq \frac{1}{2} P\left(\frac{n}{\sqrt{2}}\right)$

Two runs don't succeed with prob.

$$\leq \left(1 - \frac{1}{2} P\left(\frac{n}{\sqrt{2}}\right)\right)^2$$

≥ 1 run succeeds with:

$$P(n) \geq 1 - \left(1 - \frac{1}{2} P\left(\frac{n}{\sqrt{2}}\right)\right)^2$$

Solves to $P(n) \geq \Omega\left(1/\log n\right)$

So if do $O(\log^2 n)$ runs then

prob. of success $\geq 1 - \frac{1}{\text{Poly}(n)}$

$\Rightarrow$ Total running time $O(n^2 \log^3 n)$

Corollary: For any graph $G=(V,E)$,

$\leq \binom{n}{2}$ cuts of min size.

& for any specific one, find it with prob. $\geq 1/\binom{n}{2}$ by Karger's original alg.

So can find all in poly(n) time.

If prob. $\leq \frac{1}{n^{10}}$ of not finding min cut $(S,\bar{S})$

then prob. $\leq \frac{1}{n^{8}}$ of not finding all min cuts.

$$\Pr(\text{not finding one of the min-cuts})$$

$$= \Pr\left(\bigcup_{i=1}^{\binom{n}{2}} \text{not finding } i^{th} \text{ min-cut}\right)$$

$$\leq \sum_{i=1}^{\binom{n}{2}} \Pr(\text{not finding } i^{th} \text{ min-cut})$$

$$\leq \binom{n}{2} \frac{1}{n^{10}} \leq \frac{1}{2n^{8}} \leq \frac{1}{n^{8}}$$