Probabilistic method:

Let $A =$ good event & $B =$ bad event.

For example, for CNF formula $f$, for a random assignment let $E$ be the event that $f$ is satisfied.

If we prove that $\Pr(A) > 0$ (or equivalently $\Pr(B) < 1$),
then this shows there exists a satisfying assignment & so $f$ is satisfiable.

A natural approach is to break $A$ into smaller events $A_1, \ldots, A_n$ where $A = \bigcap_{i=1}^{n} A_i$.

If these $A_i$ are independent of each other & for each $i$, $\Pr(B_i) \leq p$ (equivalently $\Pr(A_i) \geq 1 - p$) then we have that:

\[ \Pr(A) = \Pr(\bigcap_{i=1}^{n} A_i) \geq (1 - p)^n > 0 \text{ if } p < 1. \]

equivalently,

\[ \Pr(B) = 1 - \Pr(A) < 1 - (1 - p)^n < 1 \text{ if } p < 1. \]

= $\Pr(B_i)$
Often we don't have independence between the events.
LLL = Lovász Local Lemma allows some dependencies.

**Definition:** For events \( E \) & \( F \),

\( E \) is mutually independent of \( F \) if:

\[
Pr(E|\neg F) = Pr(E).
\]

Moreover, for a set of events \( \{F_i\} \):

Then \( E \) is mutually independent of the set \( \{F_i\} \) if for all subsets \( C \subset \{F_i\} \):

\[
Pr(E|\neg C) = Pr(E).
\]

**Lovász Local Lemma:**

Let \( B_1, \ldots, B_n \) be a set of "bad" events where for each \( i \):

\[
Pr(B_i) \leq p < 1
\]

& \( B_i \) is mutually independent of all but \( \leq d \) of the other \( B_j \).

Then if \( ep(d+1) \leq 1 \),

Then:

\[
Pr(\emptyset) = Pr(\bigcap_{i=1}^n \neg B_i) = Pr(\bigcap_{i=1}^n \neg B_i) > 0
\]

(or equivalently, \( Pr(B) = 1 - Pr(\neg B) < 1 \)).

Note, a union bound says \( \sum_{i=1}^n p_i < 1 \) yields \( Pr(\emptyset) > 0 \) so this is much stronger.
Can replace $e^{p(d+1)}$ by $4pd$ which is better for $d \leq 2$ but worse as $d \uparrow$.

Here's an application of LLL:

**Lemma:** $E_k$-SAT input $f$ in which no variable appears in more than $2^{k-2} \frac{k}{k}$ clauses is satisfiable.

⇒ it doesn't say anything about the # of clauses.

**Proof:** Note, the LLL condition is $p \leq \frac{1}{e^{(d+1)}}$. We'll prove $p \leq \frac{4d^2}{k}$ which implies $\uparrow$ when $d \geq e$.

Let $B_i = \text{clause } i$ is not satisfied

$p = \Pr(B_i) = 2^{-k}$

$B_i \& B_j$ only depend on each other if they share at least one variable.

Hence, $B_i$ is dependent on $\leq k(2^{k-2})$ other clauses.

Since each of the $\frac{k}{2}$ variables in $B_i$ appears in $\leq \frac{2^{k-2}}{2}$ other clauses.

Thus, $p \leq k(\frac{2^{k-2}}{2}) = 2^{k-2} = \frac{2^k}{4}$

We have, $p = 2^{-k} \leq \frac{1}{4d} = 2^{-k}$ so LLL implies $\Pr(\text{f is satisfable}) = \Pr(\neg B_i) > 0$.
Lemma: \( \forall \text{ subset } S \subseteq \{1, \ldots, n\} \) \& any \( i \in \{1, \ldots, n\} \)

\[
\Pr(B_i | \bigcap_{j \neq i} S_j) \leq \frac{1}{d+1}
\]

Proof: Let \( m = |S| \). Induct on \( \frac{n}{m} \).

Base case: \( m = 0 \):

This we are looking at \( \Pr(B_i) \) for which we know:

\[
\Pr(B_i) \leq p \leq \frac{1}{c(d+1)} \leq \frac{1}{d+1}. \checkmark
\]

For \( m > 0 \):

Let \( D_i \) be those \( i \in \{1, \ldots, n\} \) where \( B_i \) depends on \( S_i \).

Partition \( S \) into: \( S_i = S \cap D_i \)

\& \( S_a = S \setminus S \).

Note \( |S_i| \leq 2 \) since \( B_i \) depends on \( \leq 2 \) other events.
\[ \Pr(B_i | n \text{ yes}) \]
\[ = \frac{\Pr(B_i \cap (n \text{ yes}) \cap (\neg n \text{ yes}))}{\Pr(n \text{ yes} \cap (\neg n \text{ yes}))} \]
\[ = \frac{\Pr(B_i \cap (n \text{ yes}) | (\neg n \text{ yes}))}{\Pr(n \text{ yes} | (\neg n \text{ yes}))} \]
\[ \leq \frac{\Pr(B_i | (\neg n \text{ yes}))}{\Pr(n \text{ yes} | (\neg n \text{ yes}))} \]
\[ \leq \frac{\Pr(B_i)}{\Pr(n \text{ yes} | (\neg n \text{ yes}))} \]

since \( B_i \) is indep. of \( S_2 \).

Need to bound the denominator.

\[ \Pr(B_i) \leq \frac{1}{e^{d+1}} \]
Let \( S_i = \{ j_1, \ldots, j_r \} \).

If \( r = 0 \) then \( S_i = \emptyset \) so we know that the denominator is 1 in this case.

Hence we can assume \( r > 0 \) & we know that \( r \leq \theta \) since \( |S_i| \leq \theta \) as we just pointed out.

Let \( \bar{S} = \bigcap_{l \in S_2} A_l \)

\[
\begin{align*}
\Pr\left( \bigcap_{j \in S_1} A_j \mid \bigcap_{l \in S_2} A_l \right) &= \Pr\left( \bigcap_{j \in S_1} A_j \mid \bar{S} \right) \\
&= \Pr(A_{j_1} \mid \bar{S}) \times \Pr(A_{j_2} \mid A_{j_1} \bar{S}) \times \Pr(A_{j_3} \mid A_{j_1} A_{j_2} \bar{S}) \\
&\quad \times \cdots \times \Pr(A_{j_r} \mid A_{j_1} A_{j_2} \cdots A_{j_{r-1}} \bar{S}) \\
&= \prod_{k=1}^{r} \Pr(A_{j_k} \mid \bar{S} \cap \bigcap_{l \neq k \neq j_k} A_l) \\
&= \prod_{k=1}^{r} (1 - \Pr(B_{j_k} \mid \bar{S} \cap \bigcap_{l \neq k \neq j_k} A_l)) \\
&\geq \left( 1 - \frac{1}{2^\delta} \right)^r \text{ by induction} \\
&\geq \left( 1 - \frac{1}{2^{r+1}} \right) \text{ since } r \leq \theta \\
\geq (1 - \frac{1}{2^\theta})^2 \geq \frac{1}{e} \text{ which is our desired lower bound on the denominator.}
\end{align*}
\]
Now we can prove the Lovász Local Lemma using the lemma we just proved.

We want to prove $\Pr(\bar{A}) > 0$:

$$\Pr(\bar{A}) = \Pr(\bigcap_{i=1}^{n} \bar{A}_i)$$

(by the Chain rule)

$$= \Pr(\bar{A}_1) \times \Pr(\bar{A}_2 | \bar{A}_1) \times \Pr(\bar{A}_3 | \bar{A}_1, \bar{A}_2) \times \cdots \times \Pr(\bar{A}_n | \bar{A}_1, \ldots, \bar{A}_{n-1})$$

$$= \prod_{i=1}^{n} \Pr(\bar{A}_i | \bigcap_{j<i} \bar{A}_j)$$

$$= \prod_{i=1}^{n} (1 - \Pr(B_i | \bigcap_{j<i} \bar{A}_j)$$

$$\geq (1 - \frac{1}{d+1})^n \quad \text{(by the Lemma)}$$

and

$$> 0.$$ 

but note that this is exp. small so it's unclear how to find such a solution.
Asymmetric Lovász Local Lemma:

For event $B_i$, let $D_i \subseteq \{B_1, \ldots, B_n\}$ denote the dependencies for $B_i$ (i.e., $B_i$ is independent of $\{B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_n\}$)

Note, the original form of LLL required that $|D_i| \leq d$.

**Theorem:** If there exists $x_1, \ldots, x_n \in [0,1]$ s.t. for all $i$,

$$\Pr(B_i) \leq x_i \prod_{j \in D_i} (1 - x_j)$$

then,

$$\Pr(\forall i \ B_i) = \Pr(\bigwedge_{i=1}^{n} B_i) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

**Proof:** Same proof as the original one except for in the lemma replace $\frac{1}{d+1}$ in the RHS by $x_i$.

Note, the original form follows from the asymmetric one by setting $x_i = \frac{1}{d+1}$.