CS 6550: Randomized Algorithms **Spring 2019**

Lecture 24: Counting via Sampling, Canonical Paths

April 11, 2019

Lecturer: Eric Vigoda *Scribes: Sungtae An*

24.1 Sampling and Counting

Given a graph $G = (V, E)$, let $M(G)$ = all matchings of G with any size. Then, we can consider the following two problems:

- Sampling problem: generate (sample) a matching from $\pi = \text{uniform}(M(G))$.
- Counting problem: estimate (usually with FPRAs) $|M(G)| = \#$ of matchings.

First, we can construct a Markov chain for sampling problem:

- Let $\Omega = M(G)$ = a collection of all matchings of input graph G.
- From $X_t \in \Omega$,
  1. Choose an edge $e = (v, w) \in_R E$, where $\in_R$ means that we sample an element uniformly at random.
  2. Set $X' = \begin{cases} X_t \backslash e, & \text{if } e \in X_t; \\ X_t \cup e, & \text{if } e \notin X_t. \end{cases}$
  3. If $X' \in \Omega$, then $X_{t+1} = X'$ with probability 1/2,
     otherwise $X_{t+1} = X_t$..

Note that this Markov Chain (MC) is ergodic and symmetric. Hence, the stationary distribution π for this MC is given by

$$\pi = \text{Uniform}(\Omega).$$

Moreover, we will show in the next lecture that the mixing time T_{mix} for this MC is given by

$$T_{\text{mix}} = \text{poly}(n) \quad \text{for all } G.$$

Now, we will use this sampling algorithm to approximate the counting problem, i.e., estimating $|M(G)|$. First, order the edges

$$E = \{e_1, e_2, ..., e_m\}$$

in an arbitrary order. Also, let

$$\begin{aligned} G_0 &= G \\ G_i &= G_{i-1} \backslash e_i, \quad \text{for } i > 0. \end{aligned}$$

Thus, $G_m = \varnothing$ (empty graph) on n vertices, and thus $|M(G_m)| = 1$.

Note that

$$|M(G)| = \underbrace{\frac{|M(G_0)|}{|M(G_1)|}}_{\alpha_1^{-1}} \times \underbrace{\frac{|M(G_1)|}{|M(G_2)|}}_{\alpha_2^{-1}} \times \cdots \times \underbrace{\frac{|M(G_{m-1})|}{|M(G_m)|}}_{\alpha_m^{-1}} \times \cancelto{1}{|M(G_m)|}, \tag{24.1}$$

and let

$$\alpha_i = \frac{|M(G_i)|}{|M(G_{i-1})|}.$$

Claim 24.1 $\frac{1}{2} \leqslant \alpha_i \leqslant 1.$

Proof: From Eq. (24.1),

$$|M(G)| = \frac{1}{\alpha_1 \alpha_2 \cdots \alpha_m}.$$

Note, $M(G_i) \subseteq M(G_{i-1})$ since $M \in M(G_i)$ is also in $M(G_{i-1})$, and thus $\alpha_i \leqslant 1$. Moreover, $\alpha_i \geqslant 1/2$ because $|M(G_{i-1}) \backslash M(G_i)| \leqslant |M(G_i)|$ by mapping $f : M(G_{i-1}) \backslash M(G_i) \to M(G_i)$ as $f(M) = M \backslash e_i$. Therefore, $\frac{1}{2} \leqslant \alpha_i \leqslant 1$. ■

To estimate $\alpha_i = \frac{|M(G_i)|}{|M(G_{i-1})|}$, generate samples $M_{i,1}, M_{i,2}, ..., M_{i,l}$ from Uniform($M(G_{i-1})$). Also, we can define indicator variables

$$X_{i,j} = \begin{cases} 1, & \text{if } M_{i,j} \in M(G_i); \\ 0, & \text{if not.} \end{cases}$$

Note that $\mathbb{E}[X_{i,j}] = \alpha_i$. Then, we can have an estimate for α_i by calculate

$$N = \left(\overline{X_1}\,\overline{X_2} \cdots \overline{X_m}\right)^{-1}$$

where

$$\overline{X_i} = \frac{1}{l} \sum_{j=1}^{l} X_{i,j}.$$

By Chebyshev's, for $l = O(\frac{m}{\epsilon^2})$, N is an $(1 \pm \epsilon)$-approx. for $|M(G)|$ with probability $\geqslant 3/4$.

24.2 Canonical Path

How can we bound mixing time? We can use *conductance*, which is a normalized edge expansion of MC.

A graph of MC is defined by:

- Vertices = Ω = state space
- Edges = $\{x \to y : P(x, y) > 0\}$

For set $S \subset \Omega$,

$$\Phi(S) = \Pr(X_{t+1} \notin S | X_t \in S, X_t \sim \pi) = \frac{\sum_{x \in S, y \notin S} \pi(x) P(x, y)}{\pi(S)}$$

Let

$$\Phi = \min_{S \in \Omega : \pi(S) \leqslant 1/2} \Phi(S)$$

Theorem 24.2

$$\Omega\left(\frac{1}{\Phi}\right) \leqslant T_{mix} \leqslant O\left(\frac{1}{\Phi^2} \log(\frac{1}{\pi_{min}})\right)$$

where $\pi_{min} = \min_{X \in \Omega} \pi(X)$

Proof: Easy inequality: since $\pi(S) \leqslant 1/2$, to get close to π have to at least visit $\bar{S}$. Setting $X_0 \in S, X_0 \sim \pi$, $\Phi(S)$ is the probability of leaving in 1 step, and $\frac{1}{\Phi(S)}$ is the expected number of steps to leave S and visit $\bar{S}$. ■

To lower bound the mixing time, find a set S with bad conductance. To upper bound the mixing time, prove that the conductance $\Phi(S) \geqslant 1/\text{poly}(n)$ for every $S \subset \Omega$. However, it doesn't give as tight bounds as coupling.

Now, we give a definition of *canonical paths* as follows.

Definition 24.3 *For every pair $I, F \in \Omega$, define a path γ_{IF} along edges in (Ω, P). γ_{IF} is a sequence of transitions from $I \to F$. We assume that a transition $M \to M' : P(M, M') = \frac{1}{|2E(G)|} = \frac{1}{2m}$.*

How many paths γ_{IF} go through edge $T = M \to M'$?

$$\begin{aligned} CP(T) &= \{(I, F) : \gamma_{IF} \ni T\} \\ &= \text{a set of paths that go though } T \end{aligned}$$

Also, we can define its congestion by

$$\rho(T) = \frac{|CP(T)|}{|\Omega|}$$

and

$$\rho = \max_T \rho(T).$$

Lemma 24.4 $\Phi \geqslant \frac{1}{2m\rho}$ *where $m = |E(G)|$.*

Lemma 24.4 implies that $T_{\text{mix}} = O\left(m^2\rho^2 \log(\frac{1}{\pi_{\min}})\right)$. We give a proof for Lemma 24.4 below.

Proof: Fix $S \subset \Omega$ where $\pi(S) \leqslant 1/2$ and thus $|S| \leqslant |\bar{S}|$ and $|\bar{S}| \geqslant \frac{|\Omega|}{2}$. Then, we bound $|E(S, \bar{S}|$ as follows. There are $|S| \times |\bar{S}|$ paths, i.e., pairs (I, F) with $I \in S, F \in \bar{S}$, and each of these crosses $S \to \bar{S}$ at least once on γ_{IF}. The number of times transition $T = S \to \bar{S} \in E(S, \bar{S})$ is crossed $\leqslant \rho|\Omega|$. Therefore, $|E(S, \bar{S}|$ is bounded by

$$|E(S, \bar{S}| \geqslant \frac{|S| \cdot |\bar{S}|}{\rho \cdot |\Omega|} \geqslant \frac{|S|}{2\rho}$$

■

24.2.1 Example: Random walk on hypercube

Now, we consider an example of random walk on hyper-cube where the state space is $\Omega = \{0, 1\}^n$.

- From $X_t \in \Omega$,
  1. Choose $i \in_R \{1, ..., n\}$ and $b \in_R 0, 1$.
  2. For all $j \neq i, X_{t+1}(j) = X_t(j)$
  3. Set $X_{t+1} = b$
- For $I, F \in \Omega$,
  - $X_0 = I$
  - for $i = 1 \to n$:
    - change X_i from $I(i) \to F(i)$

Let's consider transition $T = X \to X'$, which flips i^{th} bit. Set $E = (I(1), ..., I(i), F(i+1), ..., F(n))$.

Claim 24.5 *$E : CP(T) \to \Omega$ and E is injective where $CP(T) = \{(I, F) : \gamma_{IF} \ni T\}$*

Proof: Note that transition T agrees with F on $1^{\text{th}} i - 1$ bits, and with I on the last bits $i + 1, ..., n$. Thus, from E and T can infer F on all bits and I on all bits. In other words, given transitions $X \to X'$, we can get $I(i)$ and $F(i)$. Therefore, E is injective and clearly $E \in \Omega$. ■

Thus, $|CP(T)| \leqslant |\Omega|$, and so $\rho = O(1)$. Also, this implies $\Phi \geqslant \Omega(\frac{1}{n})$. Finally, $T_{\text{mix}} = O(n^3)$ since $\pi_{\min} = 2^{-n}$ (and $m = n$ in this problem). Note, using coupling, we got an $O(n \log n)$ bound on the mixing time.