2-SAT: Can solve in poly-time via a reduction to strongly connected components of directed graphs.

Here is a simple randomized algorithm.

1. Start with arbitrary assignment.
2. Repeat $\leq 100 n^2$ times: (unless all clauses are satisfied & then stop)
   - choose an arbitrary <u>unsatisfied</u> clause C
   - Pick a literal in C u.a.r. & satisfy that literal.
3. If we are at a satisfying assignment, output it. Else, output "unsatisfiable."

Fix a satisfying assignment, call it τ.

Let σ_t be the assignment of the alg. at round t.

Let X_t = # of variables that agree between σ_t & τ.

If $X_t = n$ then the alg. found a satisfying assignment.

$X_t \in \{0, 1, \ldots, n\}$ & is a random walk.

<u>Claim:</u> $\Pr(X_{t+1} = i+1 \mid X_t = i) \geq 1/2$.

<u>Why?</u>

Consider the unsatisfied clause C updated $X_t \to X_{t+1}$

We know τ satisfies C, hence

≥ 1 of the 2 variables in C

have opposite assignment in τ vs. σ_t

& ≤ 1 of the 2 agree in σ_t & τ.

Consider slowed down version (Y_t) where:

$Y_0 = X_0$ & $\Pr(Y_{t+1} = i+1 \mid Y_t = i) = 1/2$

"Couple" the 2 processes (X_t) & (Y_t)

so that if $Y_{t+1} = Y_t + 1$ then $X_{t+1} = X_t + 1$

or $X_t = n$.

More precisely, if $Y_t \to Y_{t+1}$ increases

then $X_t \to X_{t+1}$ chooses ≥ 1 of the

"good" variables in the chosen clause

to update.

Y_t is an unbiased random walk on $\{0, 1, \ldots, n\}$.

<u>Claim:</u> Expected time to hit = reach n is $O(n^2)$.

Let h_j = expected # of steps to reach n

starting at $X_0 = j$.

For $0<j<n$,

Note, $h_j = \frac{1}{2}h_{j-1} + \frac{1}{2}h_{j+1} + 1$

hence, $h_j - h_{j+1} = h_{j-1} - h_j + 2$

$$h_n = 0$$

$$h_0 - h_1 = 1$$

By induction, $h_j - h_{j+1} = 2j+1$

Therefore,

$$h_0 = h_0 - h_n = \sum_{i=0}^{n-1}(h_i - h_{i+1})$$

$$= \sum_{i=0}^{n-1}(2i+1) = \frac{2n(n-1)}{2} + n = n^2.$$

Finite Markov chains:

State space Ω

Consider $\Omega = \{0, 1, \ldots, N-1\}$. (often N is huge)

Think of graph on Ω

Directed edges s.t. for each $i \in \Omega$,

$$\sum_{j \in \Omega} P(i,j) = 1$$

hence, $P(i,j) = \Pr(X_{t+1} = j \mid X_t = i)$

P is $N \times N$ transition matrix

(needs to be stochastic = rows sum to 1)

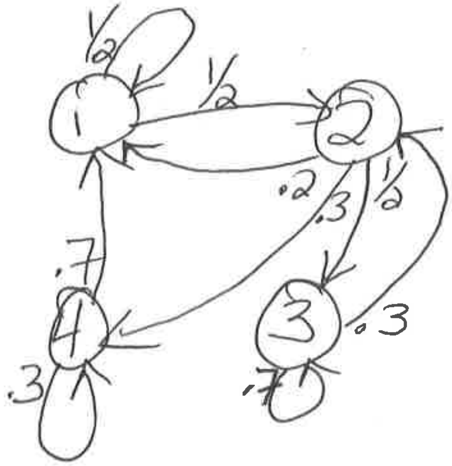
P^t is t-step probabilities:

$$\Pr(X_t = j \mid X_0 = i) = P^t(i,j).$$

If $X_0 \sim \mu_0$ then $X_t \sim \mu_t$ where $\mu_t = \mu_0 P^t$

$\mu_0 = [\quad\quad]$
row vector

Example:



$$P = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .2 & 0 & .5 & .3 \\ 0 & .3 & .7 & 0 \\ .7 & 0 & 0 & .3 \end{bmatrix}$$

Note, $$P^{20} = \begin{bmatrix} .244190 & .244187 & .406971 & .104652 \\ .244187 & .244186 & .406975 & .104651 \\ .244181 & .244185 & .406984 & .104650 \\ .244195 & .244188 & .406966 & .104652 \end{bmatrix}$$

for distribution $\pi \approx [.2442, .2442, .4070, .10465]$

$$\lim_{t\to\infty} P^t = \begin{bmatrix} \pi \\ \pi \\ \pi \\ \pi \end{bmatrix}$$

A stationary distribution satisfies: $\pi = \pi P$

i.e., invariant wrt transition matrix, like a fixed point.

(once you're in π, it stays in π)

π is an eigenvector of P with eigenvalue 1.

<u>Ergodic</u>: if $\exists t$ s.t. $\forall i,j \in \Omega$, $P^t(i,j) > 0$
(graph defined by P^t is fully-connected)

<u>Irreducible</u>: if $\forall i,j \in \Omega$, $\exists t$ s.t. $P^t(i,j) > 0$
(graph defined by P is 1 SCC)

for state $i \in \Omega$, Period of $i = \gcd\{t : P^t(i,i) > 0\}$

<u>Aperiodic</u>: Period of all $i \in \Omega$ is 1.

Ergodic $\Longleftrightarrow$ Irreducible & aperiodic.

Theorem: For a finite ergodic MC, there is a unique stationary distribution π & for all $i,j\in\Omega$,

$$\lim_{t\to\infty} P^t(i,j) = \pi(j).$$

(in words: no matter the initial distribution μ_0, $\lim_{t\to\infty} \mu_t = \pi$)

What is π?

In general need to Gaussian elimination to find it but usually $|\Omega|$ is HUGE.

If P is symmetric then $\pi=\text{uniform}(\Omega)$.

Proof: Need to verify for $\pi(i)=\frac{1}{N}$ then $\pi P=\pi$.

$$(\pi P)(i) = \sum_{k\in\Omega} \pi(k)P(k,i)$$

$$= \frac{1}{N}\sum_k P(k,i) = \frac{1}{N}\sum_k P(i,k) = \frac{1}{N} \quad \boxtimes$$

since P is symmetric since P is stochastic

Weighted symmetric:

P is reversible with respect to π if:

$$\forall i,j \in \Omega, \quad \pi(i)P(i,j) = \pi(j)P(j,i)$$

Such a π is a stationary distribution.

Proof: $(\pi P)(i) = \sum_{k\in\Omega} \pi(k)P(k,i) = \sum_k \pi(i)P(i,k) = \pi(i)\sum_k P(i,k) = \pi(i)$. ∎

Random walk on d-regular undirected graph:

for edge (i,j), $P(i,j) = P(j,i) = 1/d$

so it's symmetric & $\pi(i) = \frac{1}{n}$ for $n = |V|$.

Non-regular?

Then $\pi(i) = \frac{d(i)}{Z}$ where $d(i)$ = degree of i, $Z = \sum_j d(j) = 2m$

Check: $\pi(i)P(i,j) = \frac{d(i)}{Z}\frac{1}{d(i)} = \frac{1}{Z} = \pi(j)P(j,i)$.

What if G is directed? No idea about π! ∎

Proof that ergodic, finite MC has a stationary distribution.

Can prove using Perron-Frobenius Theorem.

Constructive Proof:

Let $h_{ij} = E[T_{ij}]$ = expected hitting time

where $T_{ij} = \min\{t : X_t = j \mid X_0 = i\}$

<u>Lemma:</u> $\pi(i) = \dfrac{1}{h_{i,i}}$ where $h_{i,i}$ = expected 1st return time for state i.

<u>Claim:</u> $h_{ij} < \infty$

<u>Proof:</u> Since P is ergodic, $\exists\, t^* \;\&\; \epsilon > 0$ s.t.

$$\forall k, \ell \in \Omega,\ P^{t^*}(k,\ell) \geq \epsilon.$$

Set $X_0 = i$.

$\Pr(X_{t^*} = j \mid X_0 = i) \geq \epsilon$.

$\& \Pr(X_{2t^*} = j \mid X_{t^*}) \geq \epsilon$.

Thus, $\Pr(\forall t \leq \ell t^*,\ X_t \neq j) \leq (1-\epsilon)^{\ell} \leq e^{-\epsilon \ell} \to 0$

$\Pr(T_{ij} > \ell t^*) \geq 1 - (1-\epsilon)^{\ell} \geq 1 - e^{-\epsilon\ell} \to 1.$ $\blacksquare$

PageRank:

Method to assign "importance" to webpages.

Graph where V = webpages
E = directed edges corresponding to hyperlinks.

Idea 1: a link is a citation, so count # of in-edges.

Idea 2: Weight outgoing links by # of hyperlinks on it,
So if page x has d outgoing links then
each gets $\frac{1}{d}$ of a citation.
Hence, it is like a random walk.

$$\pi(y) = \sum_{x : \overrightarrow{xy} \in E} \frac{1}{d(x)}$$

Idea 3: weight a page by its $\pi(x)$, hence:

$$\pi(y) = \sum_{x : \overrightarrow{xy} \in E} \frac{\pi(x)}{d(x)}$$

This corresponds to the stationary distribution of the random walk on the web graph.

But what is π? is it unique? Not necessarily because it may not be ergodic.

How to make it ergodic?

Choose $0 < \alpha < 1$,

From page $x \in V$,

with prob. α, choose a random out edge

with prob. $1-\alpha$, choose a random vertex in whole graph.

Then clearly ergodic so unique π.

But what is π?

This is the PageRank vector.

Metropolis filter:

for $x \in \Omega$, have weight $w(x) > 0$

want to design MC whose stationary distribution π satisfies $\pi(x) \propto w(x)$

in other words, $\pi(x) = \dfrac{w(x)}{\sum_{y \in \Omega} w(y)}$ $=: Z$

Choose transitions so that the graph (Ω, P) is strongly connected,

but what probabilities $P(i,j)$?

Consider transition $X_{old} \to X_{new}$:

set $P(X_{old}, X_{new}) = \min\left\{1, \dfrac{w(X_{new})}{w(X_{old})}\right\}$

Check reversibility: assume $w(X_{old}) \le w(X_{new})$

$$\pi(X_{old})P(X_{old}, X_{new}) = \frac{w(X_{old})}{Z} \times 1 = \frac{w(X_{old})}{Z}$$ since

$$\& \ \pi(X_{new})P(X_{new}, X_{old}) = \frac{\cancel{w(X_{new})}}{Z} \times \frac{w(X_{old})}{\cancel{w(X_{new})}} = \frac{w(X_{old})}{Z} =$$

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