2-SAT: can solve in poly-time via a reduction to strongly connected components of directed graphs.

Here is a simple randomized algorithm.

1. Start with arbitrary assignment.

2. Repeat \( \leq 100n^2 \) times: (unless all clauses are satisfied & then stop)
   - Choose an arbitrary unsatisfied clause \( C \)
   - Pick a literal in \( C \) u.a.r. & satisfy that literal.

3. If we are at a satisfying assignment, output it. Else, output "unsatisfiable."
Fix a satisfying assignment, call it \( \sigma \).

Let \( \sigma_t \) be the assignment of the alg. at round \( t \).

Let \( X_t = \# \) of variables that agree between \( \sigma_t \) & \( \sigma \).

If \( X_t = n \) then the alg. found a satisfying assignment.

\( X_t \in \{0, 1, \ldots, n\} \) & is a random walk.

Claim: \( \Pr(X_{t+1} = i+1 \mid X_t = i) \geq \frac{1}{2} \).

Why?

Consider the unsatisfied clause \( C \) updated \( X_t \rightarrow X_{t+1} \).

We know \( \sigma \) satisfies \( C \) hence

\( \geq 1 \) of the 2 variables in \( C \)

have opposite assignment in \( \sigma \) vs. \( \sigma_t \).

\( \& \leq 1 \) of the 2 agree in \( \sigma_t \) & \( \sigma \).
Consider slowed down version \((Y_t)\) where:
\[ Y_0 = X_0 \quad \& \quad P(Y_{t+1} = i+1 \mid Y_t = i) = \frac{1}{2} \]
"Couple" the 2 processes \((X_t)\) & \((Y_t)\)
so that if \(Y_{t+1} = Y_t + 1\) then \(X_{t+1} = X_t + 1\)
or \(X_t = n\).

More precisely, if \(Y_t \rightarrow Y_{t+1}\) increases
then \(X_t \rightarrow X_{t+1}\) chooses \(\geq 1\) of the
"good" variables in the chosen clause
to update.

\(Y_t\) is an unbiased random walk on \([0, 1, \ldots, n]^{\mathbb{R}}\).
Claim: Expected time to hit = reach \(n\) is \(O(n)\).

Let \(h_j = \text{expected # of steps to reach } n\)
starting at \(X_0 = j\).
For \(0 \leq j < n\),

\[ h_j = \frac{1}{2} h_{j-1} + h_{j+1} \]

hence, \( h_j - h_{j+1} = h_{j-1} - h_j + 2 \)

\[ h_n = 0 \]

\[ h_0 - h_1 = 1 \]

By induction, \( h_j - h_{j+1} = 2^j + 1 \)

Therefore,

\[ h_0 = h_0 - h_n = \sum_{i=0}^{n-1} (h_i - h_{i+1}) \]

\[ = \sum_{i=0}^{n-1} (2i+1) = \frac{2n(n-1)}{2} + n = \frac{n^2}{2} \]


Finite Markov chains:

State space $\mathcal{S}$
consider $\mathcal{S} = \{0, 1, \ldots, N-1\}$. (often $N$ is huge)

Think of graph on $\mathcal{S}$
directed edges s.t. for each $i \in \mathcal{S}$,

$$\sum_{j \in \mathcal{S}} p(i,j) = 1$$

hence, $p(i,j) = \Pr(X_{t+1} = j \mid X_t = i)$

$P$ is $N \times N$ transition matrix
(needs to be stochastic = row sum)

$P^+$ is $+\text{-step probabilities}$:

$$\Pr(X_{t+1} = j \mid X_t = i) = P^+(i,j)$$

If $X_0 \sim \mu_0$ then $X_t \sim \mu_+$ where $\mu_+ = \mu_0 P^+$

$$\mu_0 = \sum_{i} \mu_{0,i}$$

row vector
Example:

\[ P = \begin{bmatrix}
.5 & 5.00 \\
2 & 0.53 \\
0 & 3.70 \\
.7 & 0.03
\end{bmatrix} \]

Note, \( P^{20} = \begin{bmatrix}
.244190 & .244187 & .406971 & .104652 \\
.244187 & .244186 & .406975 & .104651 \\
.244181 & .244185 & .406984 & .104650 \\
.244195 & .244188 & .406966 & .104652
\end{bmatrix} \]

for distribution \( \Pi \propto [2442, 2442, 4070, 10465] \)

\[ \lim_{n \to \infty} P^n = \begin{bmatrix}
\Pi \\
\Pi \\
\Pi \\
\Pi
\end{bmatrix} \]

A stationary distribution satisfies: \( \Pi = \Pi P \)

i.e., invariant wrt transition matrix, like a fixed point.

(once you're in \( \Pi \), it stays in \( \Pi \))

\( \Pi \) is an eigenvector of \( P \) with eigenvalue 1.
Ergodic: if $\exists t \text{ s.t. } \forall i,j \in \mathbb{Z}, P^t(i,j) > 0$

(graph defined by $P^t$ is fully-connected)

Irreducible: if $\forall i,j \in \mathbb{Z}, \exists t \text{ s.t. } P^t(i,j) > 0$

(graph defined by $P$ is 1-cc)

for state $i \in \mathbb{Z}$, period of $i = \gcd_{i \in \mathbb{Z}} t : P^t(i,i) > 0$

Aperiodic: period of all $i \in \mathbb{Z}$ is 1.

\[
\text{Ergodic } \iff \text{Irreducible } \& \text{ Aperiodic.}
\]
Theorem: For a finite ergodic MC, there is a unique stationary distribution \( \pi \) & for all \( i, j \in \mathbb{Z} \):
\[
\lim_{t \to \infty} P^+_t(i,j) = \pi(j)
\]

(in words: no matter the initial distribution \( \mu_0 \),
\[
\lim_{t \to \infty} \mu_t = \pi
\]

What is \( \pi \)?
In general, need to Gaussian elimination to find it, but usually \(|\mathbb{Z}|\) is huge.

If \( P \) is symmetric then \( \pi = \text{uniform}(\mathbb{Z}) \).

Proof: Need to verify for \( \pi(i) = \frac{1}{N} \) then \( \pi P = \pi \).
\[
(\pi P)(i) = \sum_{k \in \mathbb{Z}} \pi(k) P(k,i)
\]
\[
= \frac{1}{N} \sum_{k \in \mathbb{Z}} P(k,i) = \frac{1}{N} \sum_{k \in \mathbb{Z}} P(i,k) = \frac{1}{N}
\]

since \( P \) is symmetric since \( P \) is stochastic.
Weighted symmetric:

$P$ is reversible with respect to $\pi$ if:

$\forall i,j \in \mathbb{Z}, \quad \pi(i)P(i,j) = \pi(j)P(j,i)$

Such a $\pi$ is a stationary distribution.

Proof:

$(\pi P)(i) = \sum_{k \in \mathbb{Z}} \pi(k)P(k,i) = \sum_{k \in \mathbb{Z}} \frac{\pi(i)P(i,k)}{p(i,k)} = \pi(i)$

Random walk on 2-regular undirected graph:

For edge $(ij)$, $P(i,j) = P(j,i) = \frac{1}{d}$

So it's symmetric & $\pi(i) = \frac{1}{n}$ for $n = |V|$.

Non-regular?

Then $\pi(i) = \frac{d(i)}{Z}$ where $d(i) = \text{degree of } i$

$Z = \sum_{j} d(j) = 2m$

Check:

$\pi(i)P(i,j) = \frac{d(i)}{Z} \cdot \frac{1}{Z} = \frac{1}{Z} = \pi(j)P(j,i)$

What if $G$ is directed? No idea about $\pi$!
Proof that ergodic finite MC has a stationary distribution.

Can prove using Perron-Frobenius Theorem.

Constructive Proof:

Let $h_{ij} = E[T_{ij}] = \text{expected hitting time}$

where $T_{ij} = \min\{t \geq 0 : X_t = j | X_0 = i\}$

Lemma: $\pi(i) = \frac{1}{h_{ii}}$ where $h_{ii} = \text{expected 1st return time for state } i$.

Claim: $h_{ij} < \infty$

Proof: Since $P$ is ergodic, $\exists T^* > 0 \text{ s.t. }$

$\forall k, l \in \mathbb{Z}, P^{T^*}(k, l) \geq \epsilon.$

Set $X_0 = i$.

$\Pr(X_{T^*} = j | X_0 = i) \geq \epsilon.$

$\& \Pr(X_{2T^*} = j | X_{T^*}) \geq \epsilon.$

Thus, $\Pr(T^* \leq \min \{T^*_i, X_{T^*_i} = j\} \leq (1-\epsilon)^l \leq e^{-\epsilon l} \to 0$

$\Pr(T_{ij} > l^+) \geq 1 - (1-\epsilon)^l \geq 1 - e^{-\epsilon l} \to 1.$
PageRank:
Method to assign "importance" to webpages.
Graph where \( V = \) webpages
\( E = \) directed edges corresponding to hyperlinks.

Idea 1: a link is a citation, so count \# of in-edges.

Idea 2: weight outgoing links by \# of hyperlinks on it.
So if page \( x \) has \( d \) outgoing links then each gets \( \frac{1}{d} \) of a citation. Hence, it is like a random walk.

\[ \pi(y) = \sum_{x : x \rightarrow y} \frac{1}{d(x)} \]

Idea 3: weight a page by its \( \pi(x) \), hence:

\[ \pi(y) = \sum_{x : x \rightarrow y} \frac{\pi(x)}{d(x)} \]
This corresponds to the stationary distribution of the random walk on the web graph.

But what is $\Pi$? Is it unique? Not necessarily because it may not be ergodic.

How to make it ergodic?

Choose $0 < \alpha < 1$

From page $x \in V$

with prob. $\alpha$, choose a random outedge

with prob. $1 - \alpha$, choose a random vertex in whole graph.

Then clearly ergodic so unique $\Pi$.

But what is $\Pi$?

This is the PageRank vector.
Metropolis filter:

for \( x \in \mathbb{Z} \), have weight \( w(x) > 0 \)

want to design MC whose stationary distribution \( \pi \)

satisfies \( \pi(x) \propto w(x) \)

in other words, \( \pi(x) = \frac{w(x)}{\sum_{y \in \mathbb{Z}} w(y)} = \frac{\pi(x)}{Z} \)

Choose transitions so that the graph \( E(\mathbb{Z}, \mathbb{P}) \)

is strongly connected

but what probabilities \( P_{ij} \)?

Consider transition \( X_{old} \rightarrow X_{new} \):

set \( P(X_{old}, X_{new}) = \min \left\{ 1, \frac{w(X_{new})}{w(X_{old})} \right\} \)

Check reversibility: assume \( w(X_{old}) \leq w(X_{new}) \)

\( \pi(X_{old}) P(X_{old}, X_{new}) = \frac{w(X_{old})}{Z} \times 1 = \frac{w(X_{old})}{Z} \)

\( \pi(X_{new}) P(X_{new}, X_{old}) = \frac{w(X_{new})}{Z} \times \frac{w(X_{old})}{w(X_{new})} = \frac{w(X_{old})}{Z} \)