Today: Given a graph $G = (V, E)$, sample uniformly at random from $M(G)$ where $M(G)$ = all matchings of $G$.

MC on matchings:

From $X_t \in \mathcal{M}$,

1. Choose $e = (v, w) \in E$.
2. If $ee X_t$, $X' = X_t \cup e$.
3. If $v \& w$ are unmatched in $X_t$, then $X' = X_t \cup ve$.
4. If $v$ is unmatched & $w$ is matched, say $(w, z) \in X_t$, then $X' = X_t \cup ve \setminus (w, z)$.
5. Set $X_{t+1} = X'$ with prob. $\frac{1}{2}$ & $X_{t+1} = X_t$ otherwise.

Ergodic & Symmetric: stationary is $\pi = \text{uniform}(M(G))$.

Note, step 4 is not necessary. (can achieve by 2+3).
What if each $M \in \mathcal{M}(G)$ has a weight $w(M) > 0$ and we want
\[ \pi(M) = \frac{w(M)}{Z} \text{ where } Z = \sum_{M} w(M). \]

Change (5) to:

Set $X_{t+1} = X$ with prob. $\min\left\{ 1, \frac{w(X)}{w(X_{t+1})} \right\}$

This is the Metropolis filter & can easily check that
\[ \pi(M) P(m, m') = \pi(m') P(m, m') \]
for this

For example, if edges have weights $\mathcal{A}(e) > 0$ and we can assign $w(M) = \mathcal{A}(M) = \prod_{e \in M} \mathcal{A}(e)$. 
Then let $p = \max \{ p(T) \}$

Since $P(m, j) = \Theta(\frac{1}{m})$, where $m \to \infty$

where $\phi(T) = \frac{3 \zeta(3)}{4 T^3}$, $T \to \infty$ for the path $\phi(T)$

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For every pair $I \leftrightarrow J$ define a path $p(IJ)$

For transition $T = M \to M - 1$

Let $p_{max} = \Theta(\frac{1}{m})$

Canonical Paths

How to prove rapid mixing?}

$\zeta(3)$

$\zeta(3)$
What are the canonical paths $X_F$?

First, order the vertices $V = \{ v_1, \ldots, v_n \}$.
For $I, F \in \mathbb{R}$, look at $I \oplus F$:

- Each component in $I \oplus F$ is either an alternating cycle, an alternating path, or an augmenting path.

Can "unwind" each such component by seq. of transitions:

- Possibly remove an edge of $F$, and then sequence of slides (step 4), and finally an add (possibly).

So order the components by lowest # vertex in each, and then unwind in order.
Fix $T = M \to M'$.

How many $I, F$ have $X_{IF} \in T$? ($X_{IF}$ goes through $T$)

Define a mapping $M : cp(T) \to 2 \times E$
which is injective (can invert uniquely),
and hence $|cp(T)| \leq |2 \times m| \leq O(m)$.

Suppose $M' = M \cup e \Leftrightarrow e$ & thus let $\hat{M} = M 
\uparrow \uparrow 
\uparrow 
$ = $M \cup e$
$\uparrow 
$ = $M', e$

Let $N = (INF) \cup (\text{I} \& \text{F} \setminus \hat{M})$

$\uparrow$
Common edges

$\uparrow$
Difference of $T$
on $\text{I} \& \text{F}$

From $N$ plus $T = M \to M'$ (possibly the 1st edge
removed or the current cycle)

We can uniquely determine $I \& F$. 
Let \( P \) be the set of perfect matchings.

Can we sample from \( \mu = \text{uniform}(P) \)?

Is there a Markov chain?

Need to be connected over \( P \).

Not sure how to do it.

Let \( N = \text{near-perfect matchings} \).

\( N \) have 2 holes = unmatched vertices.

Let \( Z = P \cup N \).

Can we design a chain that's ergodic over \( Z \)?

Yes, same chain as before, restricted to \( Z \) (instead of \( \mu(G) \)).
Can we prove rapid mixing?
Are the canonical paths valid?
For $I \in P$, $F \in P$, the path $I \rightarrow F$ stays in $\mathcal{R}$
Since $\leq 2$ holes at any time.

For $I \in N$, $F \in P$, can unwind the augmenting path
first & then the alternating cycles &
this path stays in $\mathcal{R}$.
Similarly for $I \in P$, $F \in N$.
What about $I \in N$ & $F \in N$?

Choose a random $P \in \mathcal{P}$, and go $X \rightarrow P$ then $F$.
How much does this increase the congestion?
For $A \in N$, $B \in P$, the expected increase is
\[
\frac{|N|}{|P|} \leq \text{choice of the other endpoint} \leq \text{choice of } B.
\]

So we need $\alpha = \frac{|N|}{|P|} \leq \text{Poly}(n)$.
Hence, if $\xi = \frac{|N|}{|\mathcal{P}|} \leq \text{Poly}(n)$ then the mixing time is $\text{Poly}(n)$.

But stationary distribution is $\mu = \text{uniform}(\text{NUF})$ and we are interested in $\mathcal{P}$? With prob. $\frac{1}{\xi}$, a sample from $\mu$ is in $\mathcal{P}$ & if it is in $\mathcal{P}$ it is $\text{uniform}(\mathcal{P})$.

Suppose $\deg(v) \geq \frac{n}{2}$ \forall $v \in V$.

Then $\alpha = O(n^2)$ & so we can generate a random perfect matching in $\text{poly}(n)$ time for dense graphs.
General bipartite graphs?

Corresponds to permanent of 0-1 matrix \( A \):

\[
\text{Per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^n A_{\sigma(i)}
\]

Permutations of \( 1, \ldots, n \)

Give each matching \( Me \subseteq \mathcal{U} \subseteq \mathcal{R} \) a weight based on its hole pattern.

Let \( V = \mathcal{U} \cup \mathcal{R} \). \( |L| = |R| = n \).

For \( yeE, \in G \)

let \( w(y,z) = \frac{|P|}{|N(y,z)|} \)

where \( N(y,z) \) = \(#\) of near-perfect matchings with \( y \& z \) as the unmatched
For $\text{Men}(y,z)$, let $w(M) = w(y,z) = \frac{19}{\ln(y,z)}$.

For $\text{Men}$, let $w(M) = 1$.

Note, $w(N(y,z)) = \sum_{\text{Men}(y,z)} w(y,z) = 181/\ln(y,z)$.

$w(\emptyset) = 181$.

Hence, if the stationary distribution is prop. to $w()$ then $\pi(\emptyset) = \frac{1}{e^{3}+1}$.

Turns out that with these weights then $T_{\text{mix}} = \text{Poly}(n)$.

Why?

Need $w(I)w(F) \leq w(I)w(F)$

Consider $I,F \in \emptyset$.

Then $T = M \rightarrow M$, where $\text{Men}(y,z)$ & $\text{Men}(a,b)$ where $(a,z) \in E, (b,y) \in E$.

Hence, one can show $w(y,z)w(a,b) \leq 1$ when $S$. 

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But how to get these weights?

Suppose we have weights $\hat{\omega}$ where:

\[
\frac{w(y,z)}{2} \leq \hat{\omega}(y,z) \leq 2w(y,z)
\]

Run MC wrt $\hat{\omega}$ to get $\hat{\tau}$.

Note, $\frac{\tau(y,z)}{\tau(y)} = \frac{|N(y,z)|\hat{\omega}(y,z)}{\hat{\omega}(y)}$ 
& $\hat{\tau}(\theta) = 1_\theta$

Thus,

\[
\frac{\tau(y,z)}{\tau(\theta)} = \frac{|N(y,z)|\hat{\omega}(y,z)}{\hat{\omega}(y)}
\]

Therefore,

\[
\omega(y,z) = \hat{\omega}(y,z) \cdot \frac{\tau(\theta)}{\tau(y,z)}
\]

So rough weights $\hat{\omega}$ can be boosted to close-approx. weights $\omega$.

Since can use samples from $\hat{\tau}$ to estimate this ratio.
For input graph $G = (L, U, R, E)$

assign activities $\lambda(y,z) = \frac{1}{2} \sum_{i=1}^{\lambda_i} \lambda_i$ if $(y,z) \in E$

Start with $\lambda_0 = 1$ so that it's known

slowly go from $\lambda_0 = 1 > \lambda_1 > \ldots > \lambda_n$ to 0

so that the final graph $\lambda_n \cong G$.

Set $\lambda_i = \lambda_{i-1} e^{-\frac{1}{2n}}$ so that for matching $M$, $\lambda_i(M) \geq \frac{\lambda_{i-1}(M)}{2}$

then $N = O(n^2 \log n)$

& $w_{i-1}$ is a 2-approx of $w_i$

& we can use samples from $\Pi \lambda_i w_{i-1}$ to get a good approx of $w_i$.  