Convex body $K \subseteq \mathbb{R}^n$ in n-dimensions.

Given K by an oracle (membership):

For $x \in \mathbb{R}^n$, the oracle states whether $x \in K$ or $x \notin K$.

Let $B_n = B_n(0,1)$ denote the unit ball at the origin in n-dimensions.

Assume $B_n \subseteq K$ & $K \subseteq D B_n$ for a given $D > 1$, So D is the "diameter."

Measure running time by # of oracle calls & assume infinite precision since working with real numbers.

__Sampling problem:__ Generate a point x u.a.r. from K.

__Counting problem:__ Compute $\text{vol}(K)$.

This is #P-hard [Dyer-Frieze '88], hence can't hope to compute exactly in poly-time So aim for an FPRAS.

First FPRAS presented by [Dyer, Frieze, Kannan '91]. Latest is [Lovász, Vempala '03] which gives an $O^*(n^4)$ time (i.e., oracle calls) for volume estimation.

Volume via sampling:

Given a sampling algorithm, how do we estimate $\mathrm{vol}(K)$?

Know $B_n \subseteq K \subseteq D B_n$.

Define sequence $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m$ for $m = \Theta(n \log D)$,

where $K_0 = B_n$ & $K_m = K$.

Also want that: $\mathrm{vol}(K_i) \leq 2\,\mathrm{vol}(K_{i-1})$, i.e., $\frac{1}{2} \leq \frac{\mathrm{vol}(K_{i-1})}{\mathrm{vol}(K_i)} \leq 1$.

Hence, if we generate $x \in_R K_i$ then $\Pr(x \in K_{i-1}) = \frac{\mathrm{vol}(K_{i-1})}{\mathrm{vol}(K_i)} \geq \frac{1}{2}$.

Simplest approach: $K_i = K \cap (2^{i/n} B_n)$.

Claim: $\mathrm{vol}(K_i) \leq 2\,\mathrm{vol}(K_{i-1})$.

Proof: $K_i = 2^{i/n} B_n \cap K \subseteq 2^{1/n}\left(2^{(i-1)/n} B_n \cap K\right) = 2^{1/n} K_{i-1}$. ∎

By Chebyshev's inequality, need $O(m/\epsilon^2)$ samples per K_i & $O(m^2/\epsilon^2)$ in total to estimate $(1 \pm \epsilon)\mathrm{vol}(K)$ with prob. $\geq 3/4$ using:

$$\mathrm{vol}(K) = \frac{\mathrm{vol}(K_m)}{\mathrm{vol}(K_{m-1})} \times \frac{\mathrm{vol}(K_{m-1})}{\mathrm{vol}(K_{m-2})} \times \cdots \times \frac{\mathrm{vol}(K_1)}{\mathrm{vol}(K_0)} \times \mathrm{vol}(B_n)$$

& $\mathrm{vol}(B_{2n}) = \pi^n / n!$

Better approach in [Lovasz-Vempala '03]: Use exponential distributions.

How to sample?

<u>Ball walk:</u> Two versions - lazy & speedy.

Let $\delta = \Theta\left(\frac{1}{\sqrt{n}}\right)$

<u>Lazy walk:</u> (this is the algorithm used)

From $X_t \in K$:

1. With prob. $\frac{1}{2}$, set $X_{t+1} = X_t$.
2. Else:
   - Choose X' uniformly at random from $B(X_t, \delta)$
   - If $X' \in K$, set $X_{t+1} = X'$
     else $X_{t+1} = X_t$.

<u>Speedy walk:</u> (just used in the proof)

From $X_t \in K$:

1. Choose X_{t+1} uniformly at random from $B(X_t, \delta) \cap K$.

How do you implement speedy walk?

For $x \in K$, let $\ell(x) = \frac{\mathrm{vol}(B(x,\delta) \cap K)}{\mathrm{vol}(B(x,\delta))}$

To implement speedy: do lazy until get a point $x \in K$; this takes $1/\ell(x)$ steps in expectation.

What's the invariant measure (i.e., stationary distribution)?

Statespace is uncountably infinite so standard results don't apply.

For <u>lazy walk</u>, invariant measure μ is uniform:

$$\text{for } A \subseteq K, \quad \mu(A) = \frac{\mathrm{Vol}_n(A)}{\mathrm{Vol}_n(K)}$$

(This is unique, see Thm. 2.1 in [Vempala '05] MSRI notes or Section 1 in [Lovász-Simonovitz '93]

For <u>speedy walk</u>,

$$\mu(A) = \frac{\int_A \ell(x)\,dx}{L} \quad \text{where } L = \int_K \ell(x)\,dx$$

For <u>speedy walk</u>, $\Phi \geq \frac{c\delta^2}{D^2 n}$ for constant $c > 0$.

This is the "right" answer b/c think of an unbiased walk on the integers $0, 1, \ldots, D$, which takes $\Theta(D^2)$ time to mix. And here the walk moves $\approx \frac{\delta}{\sqrt{n}}$ in 1 direction, and hence rescale to $\frac{D\sqrt{n}}{\delta}$ & this becomes $O\left(\frac{D^2 n}{\delta}\right)$.

This implies rapid mixing for <u>lazy walk</u> from a "warm start" which is an initial distribution ω where $\forall A \subseteq K$:

$$\cancel{\frac{\mu(A)}{2} \leq} \; \omega(A) \leq 2\mu(A),$$

which we obtain by using a random sample from K_{i-1} as the initial distribution on K_i.

— for a warm start unlikely to start in a "corner" & unlikely to ever get into a corner.

To bound conductance look at the following generalization:

let $$\lambda = \min_{\substack{f: K \to \mathbb{R}: \\ f \text{ is not constant}}} \frac{\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu f}$$

when $f = \mathbb{1}(S) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$ for $S \subset \cancel{K}$

then $$\frac{\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu f} = \Phi(S).$$

for discrete-time MC on (Ω, P, π):

$$\begin{aligned}
\mathrm{Var}_\pi f &= E_\pi f^2 - (E_\pi f)^2 \\
&= \sum_{x \in \Omega} \pi(x) f(x)^2 - \left(\sum_{x \in \Omega} \pi(x) f(x) \right)^2 \\
&= \sum_{x \in \Omega} \pi(x) f(x)^2 \sum_{y \in \Omega} \pi(y) - \sum_x \pi(x) f(x) \sum_y \pi(y) f(y) \\
&= \sum_{x,y \in \Omega} \pi(x) \pi(y) \left(f(x)^2 - f(x) f(y) \right) \\
&= \frac{1}{2} \sum_{x,y \in \Omega} \pi(x) \pi(y) \left(f(x) - f(y) \right)^2
\end{aligned}$$

Hence, $Var_\pi f$ measures how f varies globally: across all pairs x, y.

instead $\mathcal{E}_\pi(f,f)$ measures the "local" variation of f with respect to transitions P:

Thus, $$\mathcal{E}_\pi(f,f) = \frac{1}{2}\sum_{x,y\in\Omega} \pi(x)P(x,y)\big(f(x)-f(y)\big)^2$$

In fact, $\lambda = (gap)^{-1}$ where $gap = 1 - \max\{\lambda_1, |\lambda_N|\}$ for eigenvalues of P $\lambda_0 = 1 > \lambda_1 \geq \cdots \geq \lambda_N > -1$

When $f = 1(S)$ then:

for reversible MC

$$\mathcal{E}_\pi(f,f) = \frac{1}{2}\sum_{\substack{x\in S, y\notin S \\ \text{or} \\ x\notin S, y\in S}} \pi(x)P(x,y) = \sum_{x\in S, y\notin S} \pi(x)P(x,y)$$

& $$Var_\pi f = \frac{1}{2}\sum_{\substack{x\in S, y\notin S \\ \text{or} \\ x\notin S, y\in S}} \pi(x)\pi(y) = \pi(S)\pi(\bar{S}).$$

Let $\lambda = \min\limits_{f: K \to \mathbb{R}} \dfrac{\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu f}$

Note: can restrict to f s.t. $\bar{f} = E_\mu f = 0$

by shifting (doesn't change $\mathcal{E}_\mu(f,f)$ or $\mathrm{Var}_\mu f$)

For $A \subseteq K$,

$$P(x,A) = \Pr(X_1 \in A \mid X_0 = x)$$

for speedy walk,

$$P(x,A) = \frac{\mathrm{vol}(B(x,\delta) \cap A)}{\mathrm{vol}(B(x,\delta) \cap K)}$$

for $y \in B(x,\delta) \cap K$,

$$P(x,dy) = \frac{dy}{\mathrm{vol}(B(x,\delta) \cap K)}$$

For $x \in K$, let

$$h(x) = \frac{1}{2}\int_K P(x, dy)\big(f(x)-f(y)\big)^2$$
$$= \frac{1}{2\,\mathrm{vol}(B(x,\delta)\cap K)}\int_{B(x,\delta)\cap K} \big(f(x)-f(y)\big)^2\, dy$$

Assume WOLOG that $\bar{f} = E_\mu f = 0$

& then $\mathrm{Var}_\mu f = E_\mu f^2 = \int_K f^2\, d\mu$

& $\mathcal{E}_\mu(f,f) = \int_K h\, d\mu$

We want to show: for all f s.t. $\bar{f}=0$,

$$\frac{\int_K h\, d\mu}{\int_K f^2\, d\mu} \ge \lambda \text{ for } \lambda = \frac{c\delta^2}{D^2 n}.$$

This is the argument from

[KLS] = [Kannan, Lovász, Simonovits '95]

Reduction to "needle-like" case:

Suppose $\exists f$ where $\dfrac{\int_K h\,d\mu}{\int_K f^2\,d\mu} < \lambda$ & $\int_K f\,d\mu = 0$

We'll show that there is a convex $K_1 \subseteq K$ where:

$$\frac{\int_{K_1} h\,d\mu}{\int_{K_1} f^2\,d\mu} < \lambda \ \& \ \int_{K_1} f\,d\mu = 0$$

and $K_1 \subseteq [0,D] \times [0,\epsilon]^{n-1}$

for any $\epsilon > 0$ (can make arbitrarily small)

(Roughly, if there's a violating f then there's a violating f for a 1-dimensional problem)

Key geometric facts:

A: For convex $P \subset \mathbb{R}^2$,

$\exists$ point $x \in P$ s.t. every line ℓ through x,

the two sides $P^+ = P \cap \ell^+$ & $P^- = P \cap \ell^-$

have $\text{area}(P^+) \geq \frac{\text{area}(P)}{3}$ & $\text{area}(P^-) \geq \frac{\text{area}(P)}{3}$

B: For convex $P \subset \mathbb{R}^2$ of area A,

$\text{width}(P) \leq \sqrt{2A}$

where $\text{width} = \min\limits_{\substack{\ell,\ell': \\ \ell \,\&\, \ell' \text{ are parallel} \\ \text{and sandwich } P}} \text{distance}(\ell, \ell')$

Suppose for $j \geq 2$: we have convex $K_j \subseteq K$

where $K_j \subseteq [0,D]^j \times [0,\epsilon]^{n-j}$ & $\int_{K_j} f\,d\mu = 0$

& $$\frac{\int_{K_j} h\,d\mu}{\int_{K_j} f^2\,d\mu} < \lambda$$

Then we can reduce j by 1 as follows:

Base case: $j = n$:

For $2 \leq j \leq n$: these j are "fat" dimensions.

Take 1st 2 fat coordinates.

Look at projection of K_j onto them, call it P.

Take $x \in P$ from A.

Take $(n-1)$-dimensional plane G through x whose normal lies in P.

We know $\int_{K_j} f\,d\mu = 0$

So $\int_{K_j \cap G^+} f\,d\mu + \int_{K_j \cap G^-} f\,d\mu = 0$

either these are = or one is >0 & other is <0

If flip G, then signs flip.

Now rotate G & it changes continuously so at some point it changes from + to −

So they are = for some H.

Thus, $\int_{K_j \cap H^+} f\,d\mu = \int_{K_j \cap H^-} f\,d\mu = 0$

We know $\int_{K_j} h\,d\mu < \lambda \int_{K_j} f^2\,d\mu$

So either: $\int_{K_j \cap H^+} h\,d\mu < \lambda \int_{K_j \cap H^+} f^2\,d\mu$

or $\int_{K_j \cap H^-} h\,d\mu < \lambda \int_{K_j \cap H^-} f^2\,d\mu$.

Take the violating set & repeat

Each time area(P) decreases by $\geq 2/3$

So eventually area(P) $\leq \frac{1}{2}\epsilon^2$

then by B, width of S is $\leq \epsilon$.

Finally, rotate & one of these 2 dimensions is now of width $\leq \epsilon$.