Convex body $K \subseteq \mathbb{R}^n$ in $n$-dimensions.

Given $K$ by an oracle (membership):

For $x \in \mathbb{R}^n$, the oracle states whether $x \in K$ or $x \notin K$.

Let $B_n = B_n(0, 1)$ denote the unit ball at the origin in $n$-dimensions.

Assume $B_n \subseteq K \subseteq DB_n$ for a given $D > 1$.

So $D$ is the "diameter."

Measure running time by # of oracle calls & assume infinite precision since working with real numbers.

**Sampling problem:** Generate a point $x$ u.a.r. from $K$.

**Counting problem:** Compute $\text{vol}(K)$.

This is $\#P$-hard [Dyer-Frieze '88], hence can't hope to compute exactly in poly-time.

So aim for an FPRAS.

First FPRAS presented by [Dyer, Frieze, Kannan '91].

Latest is [Lovász, Vempala '03] which gives an $O^*(n^4)$ time (i.e., oracle calls) for volume estimation.
Volume via sampling:

Given a sampling algorithm, how do we estimate $\text{Vol}(K)$? Know $B_n \subseteq K \subseteq DB_n$.

Define sequence $k_0 \leq k_1 \leq \ldots \leq k_m$ for $m = \Theta(n \log B)$, where $k_0 = B_n$ & $k_m = K$.

Also want that $\text{Vol}(K_i) \leq 2 \text{Vol}(K_{i-1})$, i.e., $\frac{1}{2} \leq \frac{\text{Vol}(K_{i-1})}{\text{Vol}(K_i)} \leq 1$.

Hence, if we generate $x \in K_i$, then $\Pr(x \in K_{i-1}) = \frac{\text{Vol}(K_{i-1})}{\text{Vol}(K_i)} \leq \frac{1}{2}$.

Simplest approach: $K_i = K \cap (2^{i/n}B_n)$.

Claim: $\text{Vol}(K_i) \leq 2 \text{Vol}(K_{i-1})$.

Proof: $K_i = 2^{i/n}B_n \cap K \subseteq 2^{i/n}(2^{(i-1)/n}B_n \cap K) = 2^{i/n}K_{i-1}$.

By Chebychev's inequality, need $O\left(\frac{\text{Vol}(B)}{\epsilon^2}\right)$ samples per $K_i$ & $O\left(\frac{\text{Vol}(B)}{\epsilon^2}\right)$ in total to estimate $\approx (1 \pm \epsilon)\text{Vol}(K)$ with prob. $\geq \frac{3}{4}$ using:

$$\text{Vol}(K) = \frac{\text{Vol}(K_m)}{\text{Vol}(K_{m-1})} \times \frac{\text{Vol}(K_{m-1})}{\text{Vol}(K_{m-2})} \times \ldots \times \frac{\text{Vol}(K_2)}{\text{Vol}(K_1)} \times \text{Vol}(B_1)$$

& $\text{Vol}(B_{2n}) = \frac{\pi^n}{n!}$.
Better approach in [Lovasz-Vempala '03]: use exponential distributions.

How to sample?

**Ball walk**: Two versions - lazy & speedy.

Let $\delta = \Theta\left(\frac{1}{\sqrt{n}}\right)$

**Lazy walk**: (this is the algorithm used)

From $x \in K$:

1. With prob. $\frac{1}{2}$, set $x_{t+1} = x_t$.
2. Else:
   - Choose $x'$ uniformly at random from $B(x_t, \delta)$
   - If $x' \in K$, set $x_{t+1} = x'$
   - Else $x_{t+1} = x_t$.

**Speedy walk**: (just used in the proof)

From $x \in K$:

1. Choose $x_{t+1}$ uniformly at random from $B(x_t, \delta) \cap K$. 


How do you implement speedy walk?
For \( x \in K \), let \( l(x) = \frac{\text{Vol}(B(x, \delta))}{{\text{Vol}(B(x, \delta))}} \)

To implement speedy: do lazy until get a point \( x \in K \); this takes \( 1/l(x) \) steps in expectation.

What's the invariant measure (i.e., stationary distribution)?
State space is uncountably infinite so standard results don't apply.

For lazy walk, invariant measure \( \mu \) is uniform:
For \( A \subseteq K \), \( \mu(A) = \frac{\text{Vol}(A)}{\text{Vol}(K)} \)

(This is unique, see Thm. 2.1 in [Vempala '05] MSRI notes or Section 1 in [Lovász-Simonovitz '93]

For speedy walk,
\[ \mu(A) = \frac{\int_A l(x) \, dx}{L} \text{ where } L = \int_K l(x) \, dx \]
For speedy walk, \( \Phi \geq \frac{c \sigma^2}{D^3 n} \) for constant \( c > 0 \).

This is the "right" answer b/c think of an unbiased walk on the integers \( 0, 1, \ldots, D \), which takes \( O(D^2) \) time to mix. And here the walk moves \( \frac{\sigma}{\sqrt{n}} \) in 1 direction and hence rescale to \( \frac{D \sqrt{n}}{\sigma} \) & this becomes \( O(D^2) \).

This implies rapid mixing for lazy walk from a "warm start" which is an initial distribution \( \psi \) where \( \forall A \in K_i \):

\[ w(A) \leq 2 \mu(A), \]

which we obtain by using a random sample from \( K_{i-1} \) as the initial distribution on \( K_i \).

-for a warm start unlikely to start in a "corner" & unlikely to ever get into a corner.
To bound conductance look at the following generalization:

let \( \lambda = \min_{f: K \to \mathbb{R}, \text{ f is not constant}} \frac{\mathbb{E}_u(ff)}{\text{Var}_u(f)} \)

when \( f=1(S) = \begin{cases} 1 \text{ if } x \in S \\ 0 \text{ if } x \notin S \end{cases} \) for \( S \subseteq \Omega \)

then \( \frac{\mathbb{E}_u(ff)}{\text{Var}_u(f)} = \Theta(S) \).

for discrete-time MC on \((\Omega, P, \pi)\):

\[
\text{Var}_\pi f = \sum_{x \in \Omega} \mathbb{E}_\pi f^2 - (\mathbb{E}_\pi f)^2
= \sum_{x \in \Omega} \pi(x) f(x)^2 - \left(\sum_{x \in \Omega} \pi(x) f(x)\right)^2
= \sum_{x \in \Omega} \pi(x) f(x)^2 \sum_{y \in \Omega} \pi(y) - \sum_{x \in \Omega} \pi(x) f(x) \sum_{y \in \Omega} \pi(y) f(y)
= \sum_{x, y \in \Omega} \pi(x) \pi(y) (f(x)^2 - f(x) f(y))
= \frac{1}{2} \sum_{x, y \in \Omega} \pi(x) \pi(y) (f(x) - f(y))^2
\]
Hence, $\text{Var}_{\pi}f$ measures how $f$ varies globally across all pairs $x, y$.

Instead $\mathbb{E}_{\pi}(ff)$ measures the "local" variation of $f$ with respect to transitions $\pi$:

Thus, $\mathbb{E}_{\pi}(ff) = \frac{1}{2} \sum_{x, y \in \mathcal{S}} \pi(x) P(x, y) (f(x) - f(y))^2$

In fact, $\gamma = (\text{gap})^{-1}$ where $\text{gap} = 1 - \max_x |\lambda_x| \pi(x)$.

For eigenvalues $\gamma_0 > 1 > \gamma_1 \geq \ldots \geq \gamma_n > -1$

When $f=1(s)$ then:

$\mathbb{E}_{\pi}(ff) = \frac{1}{2} \sum_{x \in \mathcal{S}, y \in \mathcal{S}} \pi(x) P(x, y) \sum_{x \in \mathcal{S}, y \in \mathcal{S}} \pi(x) P(x, y)$

$\Delta \text{Var}_{\pi}f = \frac{1}{2} \sum_{x \in \mathcal{S}, y \in \mathcal{S}} \pi(x) \pi(y) = \pi(s) \pi(s)$. 

for reversible MC
Let \( \lambda = \min_{f : K \to \mathbb{R}} \frac{E_{uf}(f,f)}{\text{Var}_{uf}} \).

Note: can restrict to \( f \) s.t. \( f - E_{uf} = 0 \) by shifting (doesn't change \( E_{uf}(f,f) \) or \( \text{Var}_{uf} \)).

For \( A \subseteq K \),

\[
P(x,A) = \Pr(x \in A \mid x_0 = x)
\]

for speed walk,

\[
P(x,A) = \frac{\text{vol}(B(x,\delta) \cap A)}{\text{vol}(B(x,\delta) \cap K)}
\]

for \( y \in B(x,\delta) \cap K \),

\[
P(x,dy) = \frac{dy}{\text{vol}(B(x,\delta) \cap K)}
\]
For $x \in \mathcal{K}$, let
\[ h(x) = \frac{1}{2} \int_{\mathcal{K}} P(x, y) (f(x) - f(y))^2 \]
\[ = \frac{1}{2 \text{vol}(B(x, \delta) \cap \mathcal{K})} \int_{B(x, \delta) \cap \mathcal{K}} (f(x) - f(y))^2 \, dy \]

Assume WLOG that $\bar{f} = E_{\mathcal{K}} f = 0$

and then $\text{Var}_f = E_{\mathcal{K}} f^2 = \int_{\mathcal{K}} f^2 \, du$

and $E_{\mathcal{K}} (f, f) = \int_{\mathcal{K}} h \, du$

We want to show: for all $f$ s.t. $\bar{f} = 0$

\[ \frac{\int_{\mathcal{K}} h \, du}{\int_{\mathcal{K}} f^2 \, du} \geq \lambda \] for $\lambda = \frac{c\delta^2}{D^n}$

This is the argument from \cite{KLS} = \cite{Kannan, Lovász, Simonovits 95}
Reduction to "needle-like" case:

Suppose \( \frac{\int h d\mu}{\int f^2 d\mu} < \gamma \) \& \( \int f d\mu = 0 \)

We'll show that there is a convex \( k_i \leq k \) where:

\[
\frac{\int h d\mu}{\int f^2 d\mu} < \gamma \) \& \( \int f d\mu = 0
\]

and \( k_i \subseteq [0, D] \times [0, \epsilon]^{n-1} \)

for any \( \epsilon > 0 \) (can make arbitrarily small)

(Roughly, if there's a violating \( f \then there's a violating \( f \) for a 1-dimensional problem)
key geometric facts:

A: For convex \( \text{PCIR}^2 \),

\[ \exists \text{ point } x \in \text{PCIR}^2 \text{ s.t. every line } l \text{ through } x \]

the two sides \( P^+ = PN_l^+ \) \& \( P^- = PN_l^- \)

have \( \text{area}(P^+) \geq \frac{\text{area}(P)}{3} \) \& \( \text{area}(P^-) \geq \frac{\text{area}(P)}{3} \)

B: For convex \( \text{PCIR}^2 \) of area \( A \),

\[ \text{width}(P) = \sqrt{2A} \]

where \( \text{width} = \min \text{ distance}(l, l') \)

\( l, l' \)

\( l \parallel l' \) are parallel

and sandwich \( P \).
Suppose for \( j \geq 2 \) we have convex \( k_j \subset k \)

where \( k_j = [0,D_j] \times [0,E_j] \) & \( \int_{k_j} f du = 0 \)

& \[ \frac{\int_{k_j} h du}{\int_{k_j} f^2 du} < \lambda \]

Then we can reduce \( j \) by 1 as follows:

**Base case: \( j = n \)**

For \( 2 \leq j \leq n \): these \( j \) are "fat" dimensions.

Take 1st 2 fat coordinates.

Look at projection of \( k_j \) onto them call it \( P \).

Take \( x \in P \) from \( A \).

Take \((n-1)\)-dimensional plane \( G \) through \( x \) whose normal lies in \( P \).
We know \( \int_{k_j} f du = 0 \)

So \( \int_{k_j} t_1 u + \int_{k_j} t_2 u = 0 \)

Either these \( \Rightarrow \) are = or one is \( > 0 \) & other is \( < 0 \).

If flip \( G \), then signs \( \frac{\text{flip}}{\text{flip}} \).

Now rotate \( G \) & it changes continuously so at some point it changes from + to -

So they are = for some \( H \).

Thus, \( \int_{k_{j\cap H}} f du = \int_{k_{j\cap H}} f du = 0 \)

We know \( \int_{k_j} h du < \int_{k_j} f^2 du \)

So either: \( \int_{k_{j\cap H^+}} h du < \int_{k_{j\cap H^+}} f^2 du \)
or \( \int_{k_{j\cap H^-}} h du < \int_{k_{j\cap H^-}} f^2 du \).
Take the violating set & repeat
Each time area(P) decreases by $\geq \frac{2}{3}$
So eventually $\text{area}(P) \leq \frac{1}{2}e^2$
then by B, width of $S$ is $\leq e$.

Finally, rotate & one of these 2 dimensions is
now of width $\leq e$. 