Approximate counting $\Rightarrow$ Sampling

For a graph $G = (V,E)$, let $M(G) =$ all matchings of $G$.

Given a sampler for $\Pi = \text{uniform}(M(G))$, can we estimate $|M(G)|$?

We saw a Markov chain for sampling from $\Pi = \text{uniform}(M(G))$ with mixing time $T_{mix}(\mathbb{E}) = O(n^2m \log \frac{n}{\epsilon})$.

For simplicity, let's assume $\epsilon = O(1)$, which we can't actually achieve, but let's say, for any $G$, in $O(n^2m \log n)$ time we can generate a sample from $\Pi = \text{uniform}(M(G))$.

How can we use this to estimate $|M(G)|$?
(Use random matchings to estimate the number of matchings.)

Fix $G = (V,E)$.

Let $E = \{e_1, \ldots, e_m\}$ (in arbitrary order)

& $G_i = G_{i-1} \cup e_i$ for $i > 0$,

$G_0 = \emptyset =$ empty graph on $V$. 

Note, our goal is to compute $|M(G)|$.

$G = G_m \& G_0$ is trivial so $|M(G_0)| = 1$.

Notice that:

$$|M(G)| = \frac{|M(G_m)|}{|M(G_{m-1})|} \times \frac{|M(G_{m-1})|}{|M(G_{m-2})|} \times \ldots \times \frac{|M(G_1)|}{|M(G_0)|} \times |M(G)|$$

$$= |M(G_m)| \quad \text{since the other terms cancel}$$

$\& |M(G_0)| = 1$.

Let $\rho_i = \frac{|M(G_{m-i})|}{|M(G_i)|}$

So $|M(G)| = \prod_{i=1}^{m} \rho_i$.

Can we estimate $\rho_i$?

Every matching of $G_{i-1}$ is also a matching of $G_i$.

Thus $|M(G_{i-1})| \leq |M(G_i)| \& M(G_{i-1}) \subseteq M(G_i)$.

Generate a random matching $M_i$ from $T_i = \text{uniform}(M(G_i))$.

Let $Z_i = \begin{cases} 1 & \text{if } M_i \in M(G_{i-1}) \\ 0 & \text{otherwise} \end{cases}$

Then $E[Z_i] = \Pr(Z_i = 1) = \frac{|M(G_{i-1})|}{|M(G_i)|} = \rho_i$

So we have an estimator for $\rho_i$. 
Since \( M(G_{i-1}) \leq M(G_i) \) we know \( \rho_i \leq 1 \).

How small is \( \rho_i \)?

Note, \( |M(G_i) \setminus M(G_{i-1})| \leq |M(G_{i-1})| \)
by mapping \( M \) \( \to M' \subseteq M \) (i.e.,

Hence, \( |M(G_i)| = |M(G_{i-1})| + |M(G_i) \setminus M(G_{i-1})| \leq 2|M(G_{i-1})| \)

So \( \rho_i = \frac{|M(G_{i-1})|}{|M(G_i)|} \geq \frac{1}{2} \).

We'll generate \( s \) samples from \( \pi_i = \text{uniform}(M(G_i)) \)
& look at the proportion in \( M(G_{i-1}) \).

The expected number is \( = \rho_i \).

How many samples \( s \) to get a good estimate?

What's the variance of \( Z_i \)?

\[
\text{Var}(Z_i) = \mathbb{E}[(Z_i - \rho_i)^2]
\]
Since \( \mathbb{E}[Z_i] = \rho_i = \mathbb{P}(Z_i = 1) \)

\[
= \mathbb{P}(Z_i = 1)(1 - \rho_i)^2 + \mathbb{P}(Z_i = 0)\rho_i^2
\]

\[
= \rho_i(1 - \rho_i)^2 + (1 - \rho_i)\rho_i^2 = \rho_i(1 - \rho_i)
\]
Denote the $s$ samples from $\pi_i$ as $M_i^1, M_i^2, \ldots, M_i^s$.

Let $Z_i^j = 1(M_i^j \in \Omega(G_{i-1}))$ if $M_i^j \in \Omega(G_{i-1})$,

Let $\bar{Z}_i = \frac{1}{s} \sum_{j=1}^{s} Z_i^j$ be our estimator for $p_i$.

Clearly $E[\bar{Z}_i] = p_i$.

$$\text{Var}(Z_i) = p_i(1-p_i)$$

& hence $\frac{\text{Var}(Z_i)}{E[Z_i]^2} = \frac{\text{Var}(Z_i)}{p_i^2} = \frac{1}{p_i} - 1 \leq 1$

Thus, $\frac{\text{Var}(Z_i)}{p_i^2} = \frac{1}{s} \frac{\text{Var}(Z_i)}{p_i^2} = \frac{1}{s}$

Finally, we'll output $N = \left( \prod_{i=1}^{M} \bar{Z}_i \right)^{-1}$

Note: $\sum_{i=1}^{M} E[\bar{Z}_1 \bar{Z}_2 \cdots \bar{Z}_n] = \frac{1}{\Omega(G)}$. 

\[
\frac{\text{Var}(Z_1, Z_2, \ldots, Z_m)}{\left(\frac{1}{p_1} \cdots \frac{1}{p_m}\right)^2} = \frac{E[Z_1^2, \ldots, Z_m^2]}{p_1^2 \cdots p_m^2} - 1 \\
= \left(\prod_{i=1}^{m} \frac{E[Z_i^2]}{p_i^2}\right) - 1 \\
= \left(\prod_{i=1}^{m} 1 + \frac{\text{Var}(Z_i)}{p_i^2}\right) - 1 \\
= \left(1 + \frac{1}{5}\right)^m - 1 \\
\leq e^{m/5} \\
\leq \frac{e^2}{36} \quad \text{for } S = \frac{6m}{e^2}
\]

Then by Chebyshev's inequality:

\[
(1 - \frac{e}{3})p_1 \cdots p_m \leq Z_1Z_2 \cdots Z_m \leq p_1 \cdots p_m(1 + \frac{e}{3})
\]

with prob. \( \geq 1 - \left(\frac{\frac{3}{2}}{(\frac{e}{3})^2}\right) = \frac{3}{4} \) (Details on next page)

\& thus:

\[
\frac{1 - e}{p_1 \cdots p_m} \leq N = \frac{1}{Z_1 \cdots Z_m} \leq \frac{1 + e}{p_1 \cdots p_m}
\]

with prob. \( \geq \frac{3}{4} \)
Chebyshev’s inequality:

\[ \Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \]

Thus,

\[ \Pr(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2} \]

\[ \Pr \left( |\bar{Z}_1 - \bar{Z}_m - p_1 \cdots p_m| \geq \sum_{i \neq j} |p_i - p_j| \right) \]

\[ \leq \frac{\text{Var}(\bar{Z}_1, \cdots, \bar{Z}_m)}{\frac{\sigma^2}{m} \sum_{i \neq j} |p_i - p_j|^2} \leq \frac{1}{4} \]

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Interesting side note:

What if we use Chernoff bands instead of Chebyshev’s inequality?

To get an estimate of \( p_i \) within \( (1 \pm \frac{\epsilon}{36m}) \) we need \( O(m^2) \) samples & thus a total of \( O(m^3) \) samples whereas using Chebyshev’s ineq. only \( O(m^2) \) samples are needed.
The reduction: Approx. counting $\rightarrow$ sampling works for any \underline{self-reducible} problem.

Solutions for $G$ can be partitioned into solutions for $H_1, \ldots, H_e$ where each $H_i \subseteq G$ (smaller instances) for $l = \text{poly}(n)$ (we had $l = 2$).
Better approach:

For $\lambda > 0$ & graph $G = (V, E)$, let

\[ Z_\lambda = \sum_{M \in M(G)} \lambda^{\text{IM}} \]  

(partition function)

For $\lambda = 1$, $Z_1 = |\text{IM}| = \# \text{ of matchings}$.

Assume we have a sampler for $\tau_\lambda$ where $\tau_\lambda(M) = \frac{\lambda^{\text{IM}}}{Z_\lambda}$ for $M \in M(G)$.

We saw a Markov chain with mixing time:

\[ T_{\text{mix}}(\epsilon) = O(\lambda^2 n m \log(\frac{1}{\epsilon})) \]

Previously: we defined a sequence of graphs $G_0, G_1, \ldots, G_m$.

New approach: Fix $G$, define sequence $\lambda_0 < \lambda_1 < \ldots < \lambda_k$ where $\lambda_0 = 0$ & $\lambda_k = \lambda$.

This is a cooling schedule.

(If want to set $\lambda_0 = 0$ then define $0^0 = 1$ & then $Z_0 = 1$ corresponding to $\emptyset$. )
Then, \[ Z_{\lambda} = \frac{Z_{\lambda_{e}}}{Z_{\lambda_{e-1}}} \times \frac{Z_{\lambda_{e-1}}}{Z_{\lambda_{e-2}}} \times \cdots \times \frac{Z_{\lambda_{1}}}{Z_{\lambda_{0}}} \]

If \( \lambda_0 = 0 \) then \( Z_{\lambda_0} = 1 \)

& if \( \lambda_0 \neq 0 \) say \( \lambda_0 = \frac{k}{n} \) then \( Z_{\lambda_0} \leq 1 + \frac{1}{n} \)

Let \[ \alpha_i = \frac{Z_{\lambda_{i-1}}}{Z_{\lambda_i}} \quad \& \quad Z_{\lambda} = \prod_{i=1}^{l} \frac{1}{\alpha_i} \]

Can we estimate \( \alpha_i \) using samples from \( \Pi_{\lambda_i} = \Pi_{\lambda_{i-1}} \)?

For \( \lambda, \lambda' \) let \( M \sim \Pi_{\lambda} \)

& consider random variable \( W_{\lambda, \lambda'} = \left( \frac{\lambda'}{\lambda} \right)^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor} \)

Then \( \mathbb{E}_{M \sim \Pi_{\lambda}} \left[ W_{\lambda, \lambda'} \right] = \frac{1}{Z_{\lambda}} \sum_{M \in \mathcal{M}(G)} \left( \frac{\lambda'}{\lambda} \right)^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor} \lambda^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor} \)

Since \( T_{\lambda}(M) = \frac{\lambda^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor}}{Z_{\lambda}} \)

\[ = \frac{1}{Z_{\lambda}} \sum_{m} \left( \frac{\lambda'}{\lambda} \right)^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor} \lambda^{\left\lfloor \frac{\lambda}{\lambda'} \right\rfloor} \]

\[ = \frac{Z_{\lambda'}}{Z_{\lambda}} \]
Thus if we take $s$ samples from $\pi_{i-1}$ & look at $W_{\pi_{i-1}, i}$, we get an estimator for $\frac{1}{\lambda_i}$.

Set $\lambda_i = \lambda_{i-1} \left(1 + \frac{1}{4n}\right) \approx \lambda_{i-1} e^{\frac{1}{4n}}$ & then $Z_{\lambda_i} \leq 2Z_{\lambda_{i-1}}$ & the analysis from before holds.

Hence, we need $O(1)$ samples to estimate $\frac{1}{\lambda_i}$ & thus a total of $O(1^2)$ samples.

Since $\lambda_i \approx 2^{-n}$ then we need $l \approx O(n \log 2)$.

Therefore, we have $O(n^2)$ samples needed.

Possible to do with $l \approx O(\sqrt{n})$ so that $O(n)$ samples needed in total.

This has possibly huge $\frac{Z_{\lambda_i}}{Z_{\lambda_{i-1}}}$

but $\text{Var}(W_{\pi_{i-1}, i}) = O(1)$.