Review from before:

Finite set Ω (things easily generalize to countably infinite Ω)

Ergodic MC on Ω with transition matrix P & unique stationary π.

For $S \subset \Omega$, conductance $\Phi(S) = \dfrac{\sum_{x \in S, y \in \bar{S}} \pi(x) P(x,y)}{\pi(S)\pi(\bar{S})}$

$$\Phi_* = \min_{S \subset \Omega} \Phi(S)$$

$$T_{mix} = O\left(\frac{1}{\Phi_*^2} \log \frac{1}{\pi_{min}}\right)$$

Generalize to arbitrary $f : \Omega \to \mathbb{R}$.

let $\mathcal{E}_\pi(f,f) = \frac{1}{2} \sum_{x,y \in \Omega} \pi(x) P(x,y) \left(f(x) - f(y)\right)^2$

& $\mathrm{Var}_\pi f = \sum_{x \in \Omega} \pi(x) f(x)^2 - \left(E_\pi f\right)^2$

$\bar{f} = E_\pi f = \sum_{x \in \Omega} \pi(x) f(x)$

$$= \frac{1}{2} \sum_{x,y \in \Omega} \pi(x)\pi(y)\left(f(x) - f(y)\right)^2$$

$$= \sum_{x,y \in \Omega} \pi(x)\pi(y) f(x)^2 - \pi(x)\pi(y) f(x) f(y)$$

$$= \sum_{x \in \Omega} \pi(x) f(x)^2 - \left(\sum_{x \in \Omega} \pi(x) f(x)\right)^2$$

For $S \subset \Omega$, let $f = 1_S$

Namely, for $x \in \Omega$, $f(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$

Then, $\mathcal{E}_\pi(f,f) = \frac{1}{2} \sum_{\substack{x\in S, y\notin S \\ \text{or} \\ x\notin S, y\in S}} \pi(x)P(x,y) = \sum_{x\in S, y\notin S} \pi(x)P(x,y)$

for reversible MC

& $\mathrm{Var}_\pi f = \frac{1}{2} \sum_{\substack{x\in S, y\in \bar{S} \\ \text{or} \\ x\in \bar{S}, y\in S}} \pi(x)\pi(y) = \pi(S)\pi(\bar{S})$

Thus, $$\frac{\mathcal{E}_\pi(f,f)}{\mathrm{Var}_\pi f} = \Phi(S).$$

Let $$\lambda = \min_{f:\Omega\to\mathbb{R}} \frac{\mathcal{E}_\pi(f,f)}{\mathrm{Var}_\pi f}$$

Φ_* is the restriction to $f = 1_S$ for $S \subset \Omega$.

can restrict to f s.t. $\bar{f} = 0$ by shifting (Doesn't change $\mathcal{E}_\pi(f,f)$ or $\mathrm{Var}_\pi f$)

& we saw that: $T_{mix} = O\left(\frac{1}{\lambda} \log \frac{1}{\pi_{min}}\right)$

Also, it's true that $\lambda = \underset{\text{gap}}{\text{spectral}} = 1 - \lambda_*$ $\leftarrow \max\{|\lambda_2|, |\lambda_N|\}$.

Convex set $K \subset \mathbb{R}^n$ given by membership oracle.

Let $\delta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Assume: $B(0,1) \subset K \subseteq B(0,D)$.

Lazy walk:

From $X_t \in K$:

1. With prob. $\frac{1}{2}$, set $X_{t+1} = X_t$

Else:

2. Choose X' uniformly at random ~~ly~~ from $B(X_t, \delta)$
3. If $X' \in K$ set $X_{t+1} = X'$
   else $X_{t+1} = X_t$

Speedy walk:

From $X_t \in K$:

1. Choose X_{t+1} uniformly at random ~~ly~~ from $B(X_t, \delta) \cap K$

How do you implement speedy walk?

For $x \in K$, Let $\ell(x) = \dfrac{\mathrm{Vol}_n(B(x,\delta) \cap K)}{\mathrm{Vol}_n(B(x,\delta))}$

To implement speedy do $\frac{1}{\ell(x)}$ steps in expectation of Lazy in each step.

For $A \subseteq K$, $P(x, A) = \Pr(X_1 \in A \mid X_0 = x)$

for speedy walk,

$$P(x, A) = \frac{\mathrm{Vol}_n(B(x,\delta) \cap A)}{\mathrm{Vol}_n(B(x,\delta) \cap K)}$$

& for $y \in B(x,\delta) \cap K$,

$$P(x, dy) = \frac{dy}{\mathrm{Vol}_n(B(x,\delta) \cap K)}$$

Invariant measure μ:

$$\mu(A) = \int_K P(x, A)\, \mu(dx) = \int_K P(x,A)\, dx$$

For speedy walk:

$$\mu(A) = \frac{\int_A \ell(x)\, dx}{L} \quad \text{where } L = \int_K \ell(x)\, dx$$

For lazy walk:

$$\mu(A) = \mathrm{uniform}(K) = \frac{\mathrm{Vol}_n(A)}{\mathrm{Vol}_n(K)}$$

This is unique — see Theorem 2.1 in [Vempala '05 MSRI notes] & [Lovász–Simonovits '93] Section 1.

For speedy walk, $\Phi_* \geq \frac{c\delta^2}{D^2 n}$ for constant $c > 0$.

& more generally, let $\lambda = \frac{c\delta^2}{D^2 n}$

then $\min\limits_{f:K \to \mathbb{R}} \frac{\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu f} \geq \lambda$

Then, for lazy walk from a "warm start" we get rapid mixing.

Warm start means that:

~~for~~ a distribution ω is a warm start to μ if $\forall A \subseteq K$,

$$\frac{\mu(A)}{2} \leq \omega(A) \leq 2\mu(A).$$

For $x \in K$, let

$$h(x) = \frac{1}{2}\int_K P(x, dy)\,(f(x) - f(y))^2$$

$$= \frac{1}{2\,\mathrm{vol}_n(B(x,\delta)\cap K)} \int_{B(x,\delta)\cap K} (f(x) - f(y))^2\, dy$$

Assume, WLOG, $\bar{f} = E_\mu f = 0$

then $\quad \mathrm{Var}_\mu f = \int_K f^2\, d\mu$

$$\& \quad \mathcal{E}_\mu(f,f) = \int_K h\, d\mu$$

Want to show: for all f s.t. $\bar{f} = 0$,

$$\frac{\int_K h\, d\mu}{\int_K f^2\, d\mu} \geq \lambda$$

where $\lambda = \dfrac{c\delta^2}{D^2 n}$

Suppose $\exists f$ where $\dfrac{\int_K h\,d\mu}{\int_K f^2\,d\mu} < \lambda$ & $\int_K f\,d\mu = 0$

then there is a convex $K_1 \subseteq K$ where:

$$\frac{\int_{K_1} h\,d\mu}{\int_{K_1} f^2\,d\mu} < \lambda \quad \& \quad \int_{K_1} f\,d\mu = 0$$

and $K_1 \subseteq [0,D] \times [0,\epsilon]^{n-1}$
for any $\epsilon > 0$ (can make arbitrarily small).

So we can reduce it to a 1-dimensional problem.

Two key geometric facts about convex set P in 2-Dimensions.

Lemma A: For convex $P \subset \mathbb{R}^2$,

$\exists$ point $x \in P$ s.t. every line ℓ through x, the two sides $P^+ = P \cap \ell^+$ & $P^- = P \cap \ell^-$ have $\text{area}(P^+) \geq \frac{\text{area}(P)}{3}$

& $\text{area}(P^-) \geq \frac{\text{area}(P)}{3}$

Lemma B: For convex $P \subset \mathbb{R}^2$ of area A,

$$\text{width}(P) \leq \sqrt{2A}$$

$$\text{width} = \min_{\ell, \ell'} \text{distance}(\ell, \ell')$$

over pairs of parallel lines sandwiching P

ℓ, ℓ'

Suppose for $j \geq 2$: we have $\overset{\text{convex}}{\vee} K_j \subseteq K$

where $K_j \subseteq [0, D]^j \times [0, \epsilon]^{n-j}$ & $\int_{K_j} f \, d\mu = 0$

$$\& \quad \frac{\int_{K_j} h \, d\mu}{\int_{K_j} f^2 \, d\mu} < \lambda$$

then we can reduce j by 1.

Base case is $j = n$.

There are j "fat" coordinates. Take 1st 2.

Look at project of K_j onto these 2 dimensions, call it P.

Take $x \in P$ from Lemma A.

Take a $(n-1)$-dimensional plane G through x whose normal lies in P.

We know $\int_{K_j} f\,d\mu = 0$

So $\int_{K_j \cap G^+} f\,d\mu + \int_{K_j \cap G^-} f\,d\mu = 0$

either these are $=$ or one is >0 & other is <0

Flip G, then signs flip.

Now rotate G & it changes continuously so at some point it changes from $+$ to $-$

So they are $=$ for some H.

Thus, $\int_{K_j \cap H^+} f\,d\mu = \int_{K_j \cap H^-} f\,d\mu = 0.$

We know $\int_{K_j} h\,d\mu < \lambda \int_{K_j} f^2\,d\mu$

& thus either: $\int_{K_j \cap H^+} h\,d\mu < \lambda \int_{K_j \cap H^+} f^2\,d\mu$

or $\int_{K_j \cap H^-} h\,d\mu < \lambda \int_{K_j \cap H^-} f^2\,d\mu.$

Take the violating set & repeat.

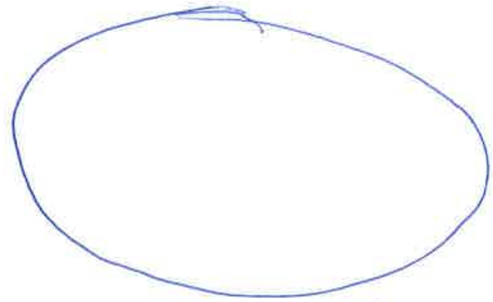
Each time the area of P decreases by $\geq \frac{2}{3}$

So eventually area$(P) \leq \frac{1}{2}\epsilon^2$

then by lemma B, width of S is $\leq \epsilon$.

Rotate & one of these 2 dimensions is now of width $\leq \epsilon$. ∎

Proof of Lemma A:



Consider a pair of parallel lines l_θ, l_θ' at angle θ where P is divided into 3 equal area parts.

Let $C_\theta = \{ l_\theta, l_\theta' : 0 \leq \theta < \pi \}$ be the collection of all such pairs, specifically the region in between.

Take a triple $C_{\theta_1}, C_{\theta_2}, C_{\theta_3}$.

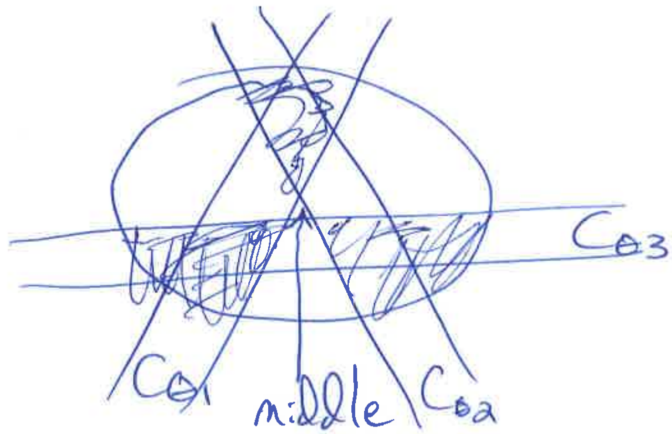
If all triples have a point in common,

~~then~~ i.e., $C_{\theta_1} \cap C_{\theta_2} \cap C_{\theta_3} \neq \emptyset$

then $\bigcap_\theta C_\theta \neq \emptyset$

& Take a point $x \in \bigcap_\theta C_\theta$ to satisfy the lemma.

Take a triple $C_{\theta_1} \cap C_{\theta_2} \cap C_{\theta_3}$ with no point in common.



Each of these 3 shaded regions has area $\geq \left(\frac{2}{3}\right)^2 = \frac{4}{9} > \frac{1}{3}$

thus there total area is > area(A) & the middle triangle is nonempty since their intersection is empty. $\Rightarrow\!\Leftarrow$.