Given a graph $G=(V,E)$,

Goal: generate a random matching of G.

Let Ω = collection of all matchings (of all sizes).

Want to sample uniformly at random from Ω in time polynomial in $n=|V|$.

Markov chain:

uniformly at random

From $X_t \in \Omega$,

1. Choose e u.a.r. from E
2. Set $X' = X_t \oplus e = \begin{cases} X_t \setminus e & \text{if } e \in X_t \\ X_t \cup e & \text{if } e \notin X_t \end{cases}$

3. If $X' \in \Omega$ then

   set $X_{t+1} = X'$ with prob. $1/2$

   Otherwise $X_{t+1} = X_t$

Ergodic & symmetric $\Rightarrow$ stationary distribution π is uniform over Ω.

We'll Prove: $T_{mix} = O(poly(n))$.

Conductance:

state space

transition matrix

stationary distribution

For a ergodic Markov chain (Ω, P, π)

For $S \subset \Omega$,

$$\text{let } \Phi(S) = \frac{\sum_{y \in S, z \in \bar{S}} \pi(y) P(y,z)}{\pi(S)}$$

Intuition: $\pi(y)P(y,z) = Pr(X_{t+1} = z \mid X_t = y, X_t \sim \pi)$

thus $\Phi(S) = Pr(X_{t+1} \in \bar{S} \mid X_t \in S, X_t \sim \pi)$

$= Pr($escape S in one step when in distribution $\pi)$

Let $\Phi_* = \min_{\substack{S \subset \Omega: \\ \pi(S) \le 1/2}} \Phi(S)$.

Theorem: $T_{mix} \le O\left(\frac{1}{\Phi_*^2} \log\left(\frac{1}{\pi_{min}}\right)\right)$

where $\pi_{min} = \min_{z \in \Omega} \pi(z)$.

Why only S where $\pi(S) \leq \frac{1}{2}$?

If S is huge, almost all of Ω, then there are few transitions out so $\Phi(S)$ is small, but this is not a real bottleneck.

Symmetrized conductance:

$$\hat{\Phi}(S) = \frac{\sum_{y \in S, z \in \bar{S}} \pi(y) P(y,z)}{\pi(S)\,\pi(\bar{S})}$$

$$\& \quad \hat{\Phi}_* = \min_{S \subset \Omega} \hat{\Phi}(S)$$

Note: for reversible chains, $\hat{\Phi}(S) = \hat{\Phi}(\bar{S})$

& WOLOG, assume $\pi(S) \leq \pi(\bar{S})$ & thus $\pi(\bar{S}) \geq 1/2$

therefore, $\Phi(S) \leq \hat{\Phi}(S) \leq 2\Phi(S)$

What if P is a random walk on a d-regular graph $G=(V,E)$?

then $\pi = \text{uniform}(V)$ so $\pi(w) = \frac{1}{n}$ for $w \in V$.

& for $(y,z) \in E$, $P(y,z) = \frac{1}{d}$ (or $1/2d$ if we do lazy version)

$$\Phi(S) = \frac{\sum_{y \in S, z \in \bar{S}} \pi(y) P(y,z)}{\pi(S)} = \frac{\frac{1}{n} \sum_{y \in S, z \in \bar{S}} \frac{\mathbb{1}((y,z) \in E)}{d}}{\frac{|S|}{n}}$$

$$= \frac{1}{d} \frac{|E(S,\bar{S})|}{|S|} = \frac{1}{d} \frac{\#\text{ of edges } S \leftrightarrow \bar{S}}{|S|}$$

<u>Intuition</u>: min-cut size = max-flow value

here we're looking at normalized cut size.

We'll define a "good" multicommodity flow.

Random walk on hypercube $\Omega = \{0,1\}^n$

$$P(x,y) = \begin{cases} \frac{1}{2} & \text{if } x = y \\ \frac{1}{2n} & \text{if } \exists i \text{ where } \begin{array}{l} x(j) = y(j) \text{ for } j \neq i \\ x(i) = 1 - y(i) \end{array} \\ 0 & \text{otherwise} \end{cases}$$

For every pair of states $x, y \in \Omega$,

we'll define a path γ_{xy} along transitions.

So $\gamma_{xy} = (z_0, z_1, \ldots, z_\ell)$

where $z_0 = x$, $z_\ell = y$

and for all $0 \leq i < \ell$, $P(z_i, z_{i+1}) > 0$.

Want to define paths with small "congestion."

For transition $u \to v$ where $P(u,v) = \frac{1}{2n}$

let $\mathcal{P}_{uv} = \{x, y \in \Omega : \gamma_{xy} \ni \overrightarrow{uv}\}$ = set of pairs x,y whose path γ_{xy} goes thru $u \to v$.

for transition $t = u \to v$ where $P(u,v) = \frac{1}{2n}$,

<u>Congestion:</u> $\rho(t) = \frac{2n}{|\Omega|} |\mathcal{P}_{uv}|$

& $\rho = \max_{t} \rho(t)$

<u>Then:</u> $T_{mix} = O\left(\rho^2 \log\left(\frac{1}{\pi_{min}}\right)\right)$

alternative, can show: $T_{mix} = O\left(\rho\, l_{max} \log\left(\frac{1}{\pi_{min}}\right)\right)$

where $l_{max} = \max_{x,y \in \Omega} |\gamma_{xy}| =$ max path length

<u>Proof:</u> Consider $S \subset \Omega$ & we'll bound

$$\hat{\Phi}(S) = \frac{1}{2n} \frac{|E(S,\bar{S})|}{|S||\bar{S}|} \times |\Omega|$$

Look at all paths between S & $\bar{S}$: $|S||\bar{S}|$ such paths.

Each crosses $E(S,\bar{S})$ ≥ 1 time.

& each such edge crossed $\leq \frac{|\Omega|}{2n} \rho$ times.

Therefore, $|E(S,\bar{S})| \geq |S||\bar{S}| \times \frac{2n}{\rho|\Omega|}$

& thus, $\frac{|E(S,\bar{S})||\Omega|}{2n|S||\bar{S}|} \geq \frac{1}{\rho}$, so $\hat{\Phi}_* \geq \frac{1}{\rho}$.

What's the canonical path γ_{xy}?

Consider $I, F \in \Omega$ & we'll define γ_{IF}.

$$\gamma_{IF} = (Z_0, Z_1, \dots, Z_n)$$

Let $Z_0 = I$.

For $i = 1 \rightarrow n$:

$$\text{for all } j \neq i, \ Z_i(j) = Z_{i-1}(j)$$

$$\& \ Z_i(i) = F(i)$$

Hence we flip the bits $i = 1 \rightarrow n$.

Note, $Z_n = F$ so we have a valid path $I \rightsquigarrow F$

& Z_{i-1}, Z_i differ at at most 1 bit

so $P(Z_{i-1}, Z_i) = \frac{1}{2n}$

(we can skip this step if $Z_{i-1} = Z_i$)

How do we bound $\rho(t)$?

Consider $t = u \to v$ where $P(u,v) = \frac{1}{2n}$

& $u \to v$ flips bit i.

We'll define a map $\eta_t : \mathcal{P}_{u,v} \to \Omega$

it will be injective & hence $|\mathcal{P}_{u,v}| \leq |\Omega|$

and thus $\rho(t) \leq O(n)$.

Recall, $t = u \to v$ flips bit i.

For $I, F \in \Omega$,
Let $E = \eta_t(I,F)$ be the following:

$$E(j) = \begin{cases} F(j) & \text{for } \cancel{j \leq i} \; j > i \\ I(j) & \text{for } \cancel{j \geq i} \; j \leq i \end{cases}$$

So it agrees with F on bits $> i$

& with I on bits $\leq i$.

Given E & $t = u \to v$

then we can uniquely decode I, F:

I agrees with E on bits $\le i$

& with u on bits $> i$

& F agrees with v on bits $\le i$

& with E on bits $> i$.

Therefore, η_t is injective.

Since $\rho = O(n)$ & $\pi_{min} = O(n)$

we get $T_{mix} = O(n^3)$.

It's a worse bound than from coupling but we can use this framework for the matchings chain next class.