Conductance and Mixing times

$M = (P, \pi, \Omega) \Rightarrow$ Markov chain

Let

$$Q(x,y) = \pi(x) P(x,y) \qquad \forall x,y \in \Omega$$

$$Q(A,B) = \sum_{x \in A} \sum_{y \in B} Q(x,y) \qquad \forall A,B \subseteq \Omega$$

Prob. of going from A to B in one step of P when starting from π

Conductance of $S \subset \Omega$:

$$\Phi(S) = \frac{Q(S,S^c)}{\pi(S)}$$

$S^c = \Omega \setminus S$

$\pi(S) = \sum_{x \in S} \pi(x)$

Prob. that a chain in eq. scapes S, given that it starts at S

~ If $\Phi(S)$ large, it is difficult/unlikely that S traps M.

Conductance:

$$\Phi_* = \min_{S : \pi(S) \leq \frac{1}{2}} \Phi(S)$$

~ Measures ability of M to scape "small" sets.

~ Small conductance $\Rightarrow$ slow mixing.

Lemma: [Jerrum, Sinclair '89] [Lawler, Sokal '88]

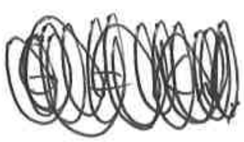
$$\frac{2}{\Phi_*^2} \log\left(\frac{1}{4\pi_{min}}\right) \geq \tau_{mix}(M) \geq \frac{1}{4\Phi^*}$$

$$\boxed{\pi_{min} = \min_{x \in \Omega} \pi(x)}$$

Example 1: Lazy - SRW on $G = (V, E)$

$$P(x,y) = \begin{cases} 1/2 & x = y \\ 0 & (x,y) \notin E \\ \dfrac{1}{2\deg(x)} & (x,y) \in E \end{cases}$$

$$\boxed{\pi(x) = \frac{\deg(x)}{2|E|}}$$



$S \subset V$

$$\Phi(S) = \frac{Q(S,S^c)}{\pi(S)} = \frac{\sum_{x\in S}\sum_{y\in S^c} \pi(x)\cdot P(x,y)}{\frac{1}{2|E|}\sum_{x \in S} \deg(x)}$$

$$= \frac{\sum_{x\in S}\sum_{y\in S^c} \frac{\cancel{\deg(x)}}{\cancel{2|E|}}\cdot\frac{1}{2\cancel{\deg(x)}}\cdot \mathbb{1}((x,y)\in E)}{\frac{1}{\cancel{2|E|}}\cdot\sum_{x\in S}\deg(x)}$$

$$\partial S = \{x \in S, y \in S^c, (x,y) \in E\}$$

$$\boxed{\Phi(S) = \frac{|\partial S|}{2\sum_{x\in S}\deg(x)}}$$

$G = K_n$

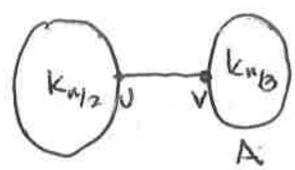
$$\Phi(S) = \frac{\cancel{|S|}(n-|S|)}{2\cancel{|S|}(n-1)} = \frac{n-|S|}{2(n-1)}$$

$$\Phi_* = \min_{S:\pi(S)\le\frac{1}{2}} \Phi(S) = \frac{n}{4(n-1)} \approx \frac{1}{4} \quad (n\to\infty)$$

$\tau_{mix} \le ?$

$$\tau_{mix} \geq \frac{1}{4\Phi^*} \approx 1$$

- In fact $\tau_{mix} \le 2$. [coupling] $\Rightarrow \left[P_r[X_2 \ne Y_2] \le \frac{1}{4} \right]$.

Example 2 L-SRW on $G =$ 

$$\Pi(A) = \sum_{x\in A} \pi(x) = \frac{1}{2|E|}\left[\sum_{x\in A\setminus v}\left[\left(\frac{n}{3}\right)-1\right]^2 + \frac{n}{3}\right] \approx \frac{n^2}{18|E|} \approx \frac{1}{5} < \frac{1}{2}$$

$$E \approx \frac{4n^2}{18} + \frac{n^2}{18} = \frac{5n^2}{18}$$

$$\Phi_* \le \Phi(A) = \frac{|\partial A|}{2\sum_{x\in A}\deg(x)} = \frac{1}{2\cdot\frac{n^2}{9}} \approx \frac{9}{2n^2}.$$

Then,
$$\boxed{\tau_{mix} \geq \frac{n^2}{18}}$$

Upper bound. (via coupling).

$X_0, Y_0 \in \Omega$

Worst case:

$X_0 \in A^c$, $Y_0 \in A$.

$$\Pr[Y_2 \in A^c] \gtrsim \frac{3}{n} \cdot \frac{3}{n} = \frac{9}{n^2}$$

independent steps from X_1 and Y_1

$$\Pr[X_2 \in A^c] \geq \left(1 - \frac{2}{3N}\right).$$

$$\Pr[X_2, Y_2 \in A^c] \gtrsim \frac{8}{n^2}$$

$$\Pr[X_4 = Y_4] \geq \Pr[X_4 = Y_4 \mid X_2 \in A^c, Y_2 \in A^c] \frac{8}{n^2}.$$

$$\Pr[X_4 = Y_4] \gtrsim \frac{6}{n^2}$$

$$\Pr[X_4 \neq Y_4] \leq 1 - \frac{6}{n^2}$$

$$\Pr[X_{4\tau} \neq Y_{4\tau}] \leq \left(1 - \frac{6}{n^2}\right)^{\tau} \leq e^{-\frac{6}{n^2}\cdot\tau}$$

$$\tau = \frac{n^2}{3}$$

$$\Pr[X_{4\tau} \neq Y_{4\tau}] \leq \frac{1}{e^2}, \text{ so } \tau_{mix} \leq \frac{n^2}{3}$$

$\boxed{\text{Ex: Work out all other possible cases for } X_0, Y_0.}$

Example 3: k-coloring on star graph.

$G =$  $|V| = n.$

Glauber dynamics:

1) Pick $w \in V$ u.a.r.
2) Assign new color to w u.a.r. from the set of available colors for w.

$\Phi_* = ?$

$$S = \{\sigma \in \Omega : \sigma(v) = 1\}$$

$$\Phi(S) = \frac{Q(S, S^c)}{\pi(S)} = \sum_{\sigma \in S} \sum_{\tau \in S^c} Q(\sigma, \tau)$$

$Q(\sigma, \tau) \neq 0$ if:

- $\sigma(v) = 1, \; \tau(v) \neq 1$
- $\sigma(w) = \tau(w) \quad \forall w \neq v$
- $\tau(w) \notin \{1, \tau(v)\} \quad \forall w \neq v.$

- \# of such pairs: $\leq (q-1)(q-2)^{n-1}.$
- For any such pair $\quad Q(\sigma, \tau) = \pi(\sigma) \cdot P(\sigma, \tau) \leq \frac{1}{|\Omega|} \cdot \frac{1}{n}$

So, $Q(S, S^c) \leq \dfrac{(q-1)(q-2)^{n-1}}{|\Omega| n}$.

and
$$\pi(S) = \frac{(q-1)^{n-1}}{|\Omega|}.$$

Then,
$$\Phi(S) \leq \frac{(q-1)(q-2)^{n-1}}{n \cdot (q-1)^{n-1}} = \frac{(q-1)^2}{n(q-2)} \left(1 - \frac{1}{q-1}\right)^n$$

$$\leq \frac{(q-1)^2}{n(q-2)} \cdot e^{-\frac{n}{q-1}}.$$

$\Phi_* \leq \Phi(S)$, and

$$t_{mix} \geq \frac{n(q-2)}{4(q-1)^2} \cdot e^{\frac{n}{q-1}}$$

Proof Lower Bound $\left(\tau_{mix} \geq \frac{1}{4\Phi_*} \right)$

For $S \subset \Omega$, let

$$\mu_S(x) = \begin{cases} 0 & x \notin S \\ \frac{\pi(x)}{\pi(S)} & x \in S \end{cases}$$

Claim 1: $\|\mu_S - \mu_S P\|_{TV} = d_{TV}(\mu_S P, \mu_S) = \Phi(S)$

Proof.

$$\Phi(S) = \sum_{x \in S} \sum_{y \in S^c} \frac{\pi(x)}{\pi(S)} \cdot P(x,y) = \sum_{x \in S} \sum_{y \in S^c} \mu_S(x) \cdot P(x,y)$$

$$= \sum_{y \in S^c} \sum_{x \in S} \mu_S(x) \cdot P(x,y) = \sum_{y \in S^c} \sum_{x \in \Omega} \mu_S(x) \cdot P(x,y)$$

$$= \sum_{y \in S^c} \mu_S P(y) = \sum_{y \in S^c} \mu_S P(y) - \mu_S(y) \qquad (1)$$

$$\|\mu_S P - \mu_S\|_{TV} = \sum_{\substack{y \in \Omega: \\ \mu_S P(y) \geq \mu_S(y)}} \mu_S P(y) - \mu_S(y) \qquad (2)$$

We show that in (1) and (2), taking the summations over S^c or over $A = \{y \in \Omega : \mu_S P(y) \geq \mu_S(y)\}$ is equivalent.

Indeed, if:

$y \in S^c \Rightarrow \mu_S P(y) \geq 0 = \mu_S(y) \Rightarrow y \in A.$

$y \notin S^c \Rightarrow y \in S$

Then,

$$\mu_S P(y) = \sum_{x \in \Omega} \mu_S(x) P(x,y) = \sum_{x \in S} \frac{\pi(x)}{\pi(S)} P(x,y)$$

$$\leq \frac{1}{\pi(S)} \cdot \sum_{x \in \Omega} \pi(x) \cdot P(x,y) \overset{(*)}{=} \frac{\pi(y)}{\pi(S)} = \mu_S(y)$$

Then, taking the summation in (1) over S^c is equivalent to summing over A instead.

This implies that $\|\mu_S P - \mu_S\|_{TV} = \Phi(S)$ as claimed ∎

(*) In class I said that this equality follows from the detailed balance conditions. Instead, it follows from the fact that $\pi(y) = \pi P(y)$.

For reversible Markov chains, the detailed balance conditions are:

$$\pi(x) P(x,y) = \pi(y) \cdot P(y,x) \quad \forall x,y \in \Omega.$$

Using Claim 1 to prove Lemma:

Property 2: Let μ, ν be distributions on Ω. Then

$$\| \mu P - \nu P \|_{TV} \leq \| \mu - \nu \|_{TV}$$

Ex. Prove property 2

Property 2 implies: $\forall u \geq 0$:

$$\| \mu_S P^{u+1} - \mu_S P^{u} \|_{TV} \leq \| \mu_S P - \mu_S \|_{TV} = \Phi(S)$$

Then

$$\| \mu_S P^{\tau} - \mu_S \|_{TV} = \left\| \sum_{u=0}^{\tau-1} \mu_S P^{u+1} - \mu_S P^{u} \right\|_{TV}$$

$$\leq \sum_{u=0}^{\tau-1} \| \mu_S P^{u+1} - \mu_S P^{u} \|_{TV}$$

$$\leq \sum_{u=0}^{\tau-1} \| \mu_S P - \mu_S \|_{TV} = \tau \cdot \Phi(S)$$

So,

$$\| \mu_S P^{\tau} - \mu_S \|_{TV} \leq \tau \cdot \Phi(S)$$

$$\| \mu_S P^{\tau} - \mu_S \|_{TV} + \| \mu_S P^{\tau} - \pi \|_{TV} \leq \tau \, \Phi(S) + \| \mu_S P^{\tau} - \pi \|_{TV}$$

$$| \pi(S^c) - \mu_S(S^c) | \leq \| \mu_S - \pi \|_{TV} \leq \tau \, \Phi(S) + \| \mu_S P^{\tau} - \pi \|_{TV}$$

$$\pi(S^c) \leq \tau \, \Phi(S) + \| \mu_S P^{\tau} - \pi \|_{TV}$$

If $\pi(S) \leq \frac{1}{2}$, then $\pi(S^c) \geq \frac{1}{2}$.

Taking $\tau = \tau_{mix}$, we get:

$$\frac{1}{2} \leq \tau_{mix} \cdot \Phi_* + \frac{1}{4},$$

and the result follows. $\blacksquare$