Colorings:

Undirected $G=(V,E)$

For vertex $w \in V$, $deg(w) = \#$ of neighbors

$$= |\{(w,z) \in E\}|$$

Maximum degree $\Delta = \max_{w \in V} deg(w)$.

Given k colors: k is an integer ≥ 2.

Proper k-coloring of G is

assignment $\sigma: V \to \{1,2,\ldots,k\}$

where $\forall (w,z) \in E$, $\sigma(w) \neq \sigma(z)$.

(So assign colors to vertices so adjacent vertices get different colors)

When $k > \Delta$ there is always a proper k-coloring.

When $k \leq \Delta$ there are graphs with no proper k-coloring

Given G & k,

let Ω = all proper k-colorings of G.

Goal: Compute $|\Omega|$ via FPRAS or sample uniformly at random from Ω.

Markov chain: (Glauber dynamics)

From $X_t \in \Omega$,

1. Choose $v \in V$ u.a.r. & $c \in \{1, \ldots, k\}$ u.a.r.
2. For all $w \neq v$, set $X_{t+1}(w) = X_t(w)$.
3. Set $X_{t+1}(v) = \begin{cases} c & \text{if } c \notin X_t(N(v)) \\ X_t(v) & \text{otherwise} \end{cases}$

Aperiodic since $P(\sigma, \sigma) > 0$.

Irreducible when $k \geq \Delta + 2$.

Idea: Let $V = \{v_1, \ldots, v_n\}$

For $\sigma \rightsquigarrow \tau$, fix vertices in order $i = 1 \to n$:

~~change~~ For v_i, if some neighbor has color $\eta(v_i)$, ~~then~~ recolor it to some available color, (at least one since $k \geq \Delta + 2$).

Then set v_i to color $\eta(v_i)$ & repeat for $i+1$.

When $k \geq \Delta+2$, it's ergodic.
Since it's symmetric, stationary dist. π is uniform (Ω).

Mixing time?
Now: when $\boxed{k > 3\Delta}$, $T_{mix} = O(n \log n)$.

Use identity coupling: For (X_t, Y_t): choose same v, c to update.

Let $A_t = \{v : X_t(v) = Y_t(v)\}$
& $D_t = \{v : X_t(v) \neq Y_t(v)\}$

$\Pr(X_t \neq Y_t) \leq E[|D_t|]$.

Let's look at $E[|D_{t+1}|]$ compared to $|D_t|$.

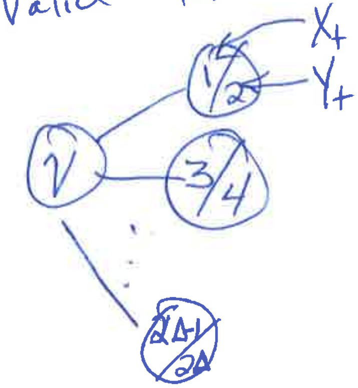
Let $a_t(v) = A_t \cap N(v)$ & $d_t(v) = D_t \cap N(v)$.

Note, $\sum_{v \in A_t} d_t(v) = \sum_{v \in D_t} a_t(v)$.

Suppose $v \in A_t$.

How many colors valid for one chain & not other?

if:



then 2Δ colors are "bad"

In general, $\leq 2d_t(v)$ colors are bad.

$$\Pr(v \in D_{t+1} \mid v \in A_t) \leq \frac{2d_t(v)}{nk}$$

Suppose $v \in D_t$,

how many colors valid for v in both chains?

$$\geq k - 2\Delta + a_t(v)$$

$$\Pr(v \in A_{t+1} \mid v \in D_t) \geq \frac{k - 2\Delta + a_t(v)}{nk}.$$

$$E\left[|D_{t+1}|\right] \le |D_t| + \sum_{v \in A_t} \frac{2d_t(v)}{nk} - \sum_{v \in D_t} \frac{(k - 2\Delta + a_t(v))}{nk}$$

$$\le |D_t| + \sum_{v \in A_t} \frac{2d_t(v)}{nk} - \sum_{v \in D_t} \frac{(k - 3\Delta + 2a_t(v))}{nk}$$

$$= |D_t| \left(1 - \frac{(k-3\Delta)}{nk}\right) \cancel{+ \frac{1}{nk} \sum_{v \in A_t} 2d_t(v) - \frac{1}{nk} \sum_{v \in D_t} 2a_t(v)}$$

$$\le |D_t| \left(1 - \frac{1}{nk}\right) \quad \text{for } k > 3\Delta$$

$$\Pr(X_t \ne Y_t) \le E\left[|D_t|\right]$$

$$\le |D_0| \left(1 - \frac{1}{nk}\right)^t$$

$$\le n e^{-t/nk}$$

$$\le \frac{1}{4} \quad \text{for } t \ge nk \log(4n)$$

So mixing time $O(nk \log n)$.

Path coupling:

Worst-case pairs: differ at 1 vertex.

For $X_t, Y_t \in \Omega$, let $H(X_t, Y_t) = |D_t| = |\{v : X_t(v) \neq Y_t(v)\}|$

Consider pairs (X_t, Y_t) where $H(X_t, Y_t) = 1$.

Let $\{z\} = X_t \oplus Y_t$.

If update $v = z$ then with prob. $\geq \frac{k-\Delta}{nk}$ they agree after

$$\Pr(X_{t+1} = Y_{t+1}) \geq \frac{k-\Delta}{nk}$$

If update $v \in N(z)$ then ≤ 2 "bad" color choices

$$\Pr(|D_{t+1}| = 2) \leq \frac{2\Delta}{nk}$$

Therefore,

$$E[|D_{t+1}| \mid H(X_t, Y_t) = 1] \leq 1 + \frac{2\Delta}{nk} - \frac{(k-\Delta)}{nk}$$

$$\leq 1 - \frac{1}{nk} \text{ for } k > 3\Delta.$$

Better coupling:

When $v \in N(z)$ (a) if $X_t \to X_{t+1}$ chooses $Y_t(z)$ then $Y_t \to Y_{t+1}$ chooses $X_t(z)$

& (b) if X_t chooses $X_t(z)$ then Y_t chooses $Y_t(z)$

(a) might be bad but (b) is blocked in both so $|D_{t+1}| = |D_t| = 1$.

Thus, $E[|D_{t+1}|] \leq 1 + \frac{\Delta}{nk} - \frac{(k-\Delta)}{nk} \leq 1 - \frac{1}{nk}$ for $k > 2\Delta$.

What about other pairs (X_t, Y_t)?

Take pair $(X,Y) \in \Omega^2$ where $H(X,Y) = \ell$.

Define sequence $W_0, W_1, \ldots, W_\ell \in \Omega$ where:

a) for all i, $H(W_{i-1}, W_i) = 1$

b) $W_0 = X$, $W_\ell = Y$.

Let $X', Y', W_0', \ldots, W_\ell'$ be the state after 1 transition.

For all i, there is a coupling for (W_{i-1}, W_i)

thus given the transition $W_{i-1} \to W_{i-1}'$

then the coupling defines $W_i \to W_i'$.

Choose a random transition for $W_0 \to W_0'$

$$\begin{array}{c} W_0 \to W_0' \\ \| \\ X \to X' \end{array}$$

then "compose" the couplings along this path $W_0 \to W_1 \to \cdots \to W_\ell$

& we get the transition

for all i, $W_i \to W_i'$

& $Y \to Y'$

Thus, we have a coupling $(X,Y) \to (X', Y')$.

How good is this coupling?

$$
\begin{aligned}
E[|D_{t+1}|] &= E[H(X_{t+1}, Y_{t+1})] \\
&\le E\left[\sum_{i=0}^{l-1} H(\omega_{i-1}', \omega_i')\right] \\
&= \sum_{i=0}^{l-1} E[H(\omega_{i-1}', \omega_i')] \\
&\le l\left(1-\frac{1}{nk}\right) \quad \text{for } k > 2\Delta \\
&= |D_t|\left(1-\frac{1}{nk}\right)
\end{aligned}
$$

Therefore,

$$
\Pr(X_t \neq Y_t) \le E[|D_t|] \le n\left(1-\frac{1}{nk}\right)^t \le \frac{1}{4}
$$

for $t \ge nk\ln(4n)$
& $k > 2\Delta$.

Path coupling: [Bubley, Dyer '97] For ergodic MC on Ω.

Let $S \subseteq \Omega \times \Omega$ s.t. (Ω, S) is connected.

For $(X,Y) \in \Omega \times \Omega$, let

$\text{dist}(X,Y) =$ length of shortest path b/w X & Y in (Ω, S).

If there exists $\beta < 1$ s.t. $\forall (X_t, Y_t) \in S$

there is a coupling $(X_t, Y_t) \to (X_{t+1}, Y_{t+1})$

where

$$E[\text{dist}(X_{t+1}, Y_{t+1})] \leq \beta \, \text{dist}(X_t, Y_t)$$

Then
$$T_{mix}(\epsilon) \leq \frac{\log(D_{max}/\epsilon)}{1-\beta}$$

where $D_{max} = \max\limits_{(X,Y) \in \Omega^2} \text{dist}(X,Y)$.