Spectral Gap & Mixing TIMES

- $\mathcal{M} = (P, \pi, \Omega)$ ergodic, reversible Markov chain.
- Eigenvalues of P: $1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{|\Omega|} \geq -1$.
- Spectral Gap: $\lambda(P) = 1 - \max_{i \geq 2} |\lambda_i|$
- If all eigenvalues non-negative, then $\lambda(P) = 1 - \lambda_2$.

Thm 1

$$\left(\lambda(P)^{-1} - 1\right) \log 2 \leq T_{mix}(\mathcal{M}) \leq \lambda(P)^{-1} \cdot \log\left(\frac{4}{\pi_{min}}\right)$$

$\pi_{min} = \min_{x \in \Omega} \pi(x)$

For simplicity, we shall assume that $\mathcal{M}$ is lazy,

i.e., $P = \dfrac{I + \hat{P}}{2}$

and show instead:

Thm 1.1 $T_{mix}(\mathcal{M}) \leq \lambda(P)^{-1} \log\left(\frac{16}{\pi_{min}}\right)$.

(*) Recall if P is lazy, then all its eigenvalues are non-negative.

Some definitions:

Take $f \in \mathbb{R}^{|\Omega|}$ [or $f : |\Omega| \to \mathbb{R}$].

$$\boxed{\Rightarrow E_\pi[f] = \sum_{x \in \Omega} \pi(x) \cdot f(x).}$$

$$\boxed{\Rightarrow \mathrm{Var}_\pi[f] = E_\pi\left[(f - E_\pi[f])^2\right]}$$

$$= \frac{1}{2} \sum_{x,y \in \Omega} \pi(x)\pi(y)\left(f(x) - f(y)\right)^2 \quad \text{[Ex. Check!]}.$$

The Dirichlet form: ("local" variance).

$$\boxed{\mathcal{E}_\pi(f,f) = \frac{1}{2} \sum_{x,y \in \Omega}' \pi(x) \cdot P(x,y)\left(f(x) - f(y)\right)^2.}$$

Lemma 2 If $\exists \alpha > 0$ s.t.

$$\forall f \in \mathbb{R}^{|\Omega|} \quad \alpha \cdot \mathcal{E}_\pi(f,f) \geq \mathrm{Var}_\pi(f),$$

then:

$$T_{mix} \leq \alpha \cdot \log \frac{16}{\pi_{min}}.$$

What is the optimal α?

Lemma 3 (Courant - Fischer Thm)

P ergodic and reversible w.r.t π. Then

$$1 - \lambda_2 = \min_{f : Var_\pi(f) \neq 0} \frac{\mathcal{E}_\pi(f,f)}{Var_\pi(f)}.$$

- Since P is lazy, all its eigenvalues are positive and so $\lambda(P) = 1 - \lambda_2$.
- Then $\alpha_{OPTIMAL} = \lambda(P)^{-1}$, and

$$T_{mix} \leq \lambda(P)^{-1} \cdot \log\left(\frac{16}{\pi_{min}}\right),$$

as claimed ∎

Note: If we consider the inner product

$$\langle f, g \rangle_\pi = \sum_{x \in \Omega} f(x) \cdot g(x) \cdot \pi(x).$$

Then, assuming $E_\pi[f] = 0$ w.l.o.g, we get:

$$\frac{\mathcal{E}_\pi(f,f)}{Var_\pi(f)} = \frac{\langle f, (I-P)f \rangle_\pi}{\langle f, f \rangle_\pi} \quad \text{[check!]}$$

Proof of Lemma 2:

$$\mathrm{Var}_\pi(Pf) \leq \mathrm{Var}_\pi(f) - \mathcal{E}_\pi(f,f) \qquad (*)$$

Before proving $(*)$, let us use it to prove Lemma 2.

Since, $\quad \mathcal{E}_\pi(f,f) \geq \frac{1}{\alpha}\cdot \mathrm{Var}_\pi(f)$ by assumption,

$$\mathrm{Var}_\pi(Pf) \leq \mathrm{Var}_\pi(f) - \frac{1}{\alpha}\mathrm{Var}_\pi(f)$$
$$\leq \left(1-\frac{1}{\alpha}\right)\mathrm{Var}_\pi(f)$$

Inductively:

$$\mathrm{Var}_\pi(P^\tau f) \leq \left(1-\frac{1}{\alpha}\right)^\tau \cdot \mathrm{Var}_\pi(f) \quad \forall f \in \mathbb{R}^{|\Omega|}.$$

Now, let

$$f_A(x) = \begin{cases} 1 & x \in A \\ 0 & \text{o.w.} \end{cases}$$

$\hookrightarrow$ indicator for the set A.

Then: $\mathrm{Var}_\pi(f_A) \leq E_\pi[f_A^2] \leq 1$, and.

$$\mathrm{Var}(P^\tau f_A) \leq \left(1-\frac{1}{\alpha}\right)^\tau \leq e^{-\tau/\alpha}.$$

Suppose $x \in \Omega$ is the initial state of the chain and take $\tau = \alpha \log\left(\frac{16}{\pi(x)}\right)$.

Then,

$$\frac{\pi(x)}{16} \geq \mathrm{Var}_\pi(P^\tau f_A) \geq \pi(x)\left(P^\tau f_A(x) - E_\pi[P^\tau f_A]\right)^2$$

$$\frac{1}{4} \geq \left|P^\tau f_A(x) - E_\pi[P^\tau f_A]\right| \quad (*)(*)$$

Now, using the fact that f_A is the indicator for set A, we get:

(a) $P^\tau f_A(x) = \sum_{y\in\Omega} P^\tau(x,y)\cdot f_A(y) = \sum_{y\in A} P^\tau(x,y) = P^\tau(x,A)$.

(b) $E_\pi[P^\tau f_A] = E_\pi[f_A] = \sum_{x\in\Omega} \pi(x)\cdot f_A(x) = \pi(A)$.

[Ex. Check!].

Plugging (a) and (b) into (*)(*), we get

$$\frac{1}{4} \geq |P^\tau(x,A) - \pi(A)|.$$

Since this holds $\forall x\in\Omega$ and $A \subseteq \Omega$,

$$\frac{1}{4} \geq \|P^\tau(x,\cdot) - \pi(\cdot)\|_{TV}$$

and so

$$T_{mix} \leq \tau = \alpha \cdot \log\left(\frac{16}{\pi_{min}}\right).$$

■

We still need to show inequality ⊛:

$$\mathrm{Var}_\pi(Pf) \le \mathrm{Var}_\pi(f) - \mathcal{E}_\pi(f,f).$$

- Recall that P is lazy: Namely, $P = \frac{I+\hat{P}}{2}$.

Let us assume w.l.o.g. that $E_\pi[f]=0$.
(Note that adding a constant to f does not change any of the terms in ⊛).

Then,

$$\mathrm{Var}_\pi(Pf) = \sum_{x\in\Omega} [Pf(x)]^2 \pi(x) = \sum_{x\in\Omega} \pi(x) \left[\sum_{y\in\Omega} P(x,y)\cdot f(y)\right]^2$$

$$= \sum_{x\in\Omega} \pi(x) \left[\sum_{y\in\Omega} \frac{I+\hat{P}}{2}(x,y)\cdot f(y)\right]^2$$

$$= \frac{1}{4}\sum_{x\in\Omega} \pi(x) \left[f(x) + \sum_{y\in\Omega} \hat{P}(x,y)\cdot f(y)\right]^2.$$

$$= \frac{1}{4}\sum_{x\in\Omega} \pi(x) \left[\sum_{y\in\Omega} \hat{P}(x,y)\,(f(x)+f(y))\right]^2.$$

$$\mathrm{Var}_\pi(Pf) \le \frac{1}{4}\sum_{x\in\Omega} \pi(x) \left(\sum_{z\in\Omega} \hat{P}(x,z)\right)\left(\sum_{y\in\Omega} \hat{P}(x,y)\,(f(x)+f(y))^2\right).$$

(Cauchy-Schwarz inequality).

$$\operatorname{Var}_\pi(P_f) \leq \frac{1}{4} \sum_{x,y\in\Omega}{}' \pi(x) \cdot \hat{P}(x,y)\left(f(x)+f(y)\right)^2 \qquad (1)$$

Now,

$$\operatorname{Var}_\pi(f) = \sum_{x\in\Omega}{}' \pi(x) f(x)^2 = \frac{1}{2}\sum_{x\in\Omega}{}' \pi(x)\cdot f(x)^2 + \frac{1}{2}\sum_{y\in\Omega}{}' \pi(y)\cdot f(y)^2$$

$$= \frac{1}{2}\sum_{x,y\in\Omega}{}' \pi(x)\cdot f(x)^2\, \hat{P}(x,y) + \frac{1}{2}\sum_{x,y\in\Omega}{}' \pi(y)\cdot f(y)^2 \cdot \hat{P}(y,x).$$

$$= \frac{1}{2}\sum_{x,y\in\Omega}{}' \pi(x)\,\hat{P}(x,y)\left(f(x)^2+f(y)^2\right). \qquad (2)$$

Then, (2) − (1) gives:

$$\operatorname{Var}_\pi(f) - \operatorname{Var}_\pi(P_f) \geq \frac{1}{4}\sum_{x,y\in\Omega}{}' \pi(x)\,\hat{P}(x,y)\left[2\left(f(x)^2+f(y)^2\right) - \left(f(x)+f(y)\right)^2\right]$$

$$\geq \frac{1}{4}\sum_{x,y\in\Omega}{}' \pi(x)\,\hat{P}(x,y)\left(f(x)-f(y)\right)^2$$

$$= \frac{1}{4}\sum_{x,y\in\Omega} \pi(x)\,(2P-I)(x,y)\left(f(x)-f(y)\right)^2.$$

$$= \frac{1}{2}\sum_{x,y\in\Omega} \pi(x)\cdot P(x,y)\left(f(x)-f(y)\right)^2 - \cancelto{0}{\frac{1}{4}\sum_{x,y\in\Omega}{}' \pi(x)\cdot I(x,y)\left(f(x)-f(y)\right)^2}$$

$$= \mathcal{E}_\pi(f,f)$$

as desired ∎

Proof of Lower Bound:

- Let $\{f_1, \dots, f_{|\Omega|}\}$ be the eigenvectors of $\hat{P}$.

- Consider the inner product:

$$\langle f, g \rangle_\pi = \sum_{x \in \Omega} f(x) \cdot g(x) \cdot \pi(x) \quad \forall f, g \in \mathbb{R}^{|\Omega|}.$$

- Consider the inner product space:

$$L_2(\pi) = \left(R^{|\Omega|}, \langle , \rangle_\pi \right)$$

$$\boxed{\text{Claim 41: } \{f_J\}_{J=1}^{|\Omega|} \text{ is an orthonormal basis of } L_2(\pi).}$$

[Direct consequences of Spectral thm for symmetric matrices. Check!!]

Orthonormal basis:

- $\langle f_J, f_i \rangle = 0 \quad \forall J \neq i$ [orthogonality]
- $\langle f_i, f_i \rangle = 1 \quad \forall i$ [normality]
- $\forall g \in \mathbb{R}^{|\Omega|}: \quad g = \sum_{J=1}^{\ell} a_J \cdot f_J$ [basis]

$$a_J = \langle g, f_J \rangle_\pi.$$

Lower bound :

$$\left(\lambda(\hat{P})^{-1} - 1 \right) \log 2 \leq \tau_{mix}.$$

Note that : $\displaystyle\sum_y{}' \pi(y) \cdot f_J(y) = \langle f_J, \vec{1} \rangle_\pi = 0$ $\left[\begin{array}{l} \text{Since } f_1 \\ \text{orthogonal} \\ \text{to } f_J \end{array}\right]$

Then,

$$\left| \lambda_J^{\tau} \cdot f_J(x) \right| = \left| \hat{P}^{\tau} f_J(x) \right| = \left| \sum_y{}' P^{\tau}(y,x) \cdot f_J(y) \right|$$

$$= \left| \sum_y{}' P^{\tau}(x,y) \cdot f_J(y) - \sum_y{}' \pi(y) \cdot f_J(y) \right|$$

$$\leq \sum_y{}' \left| P^{\tau}(x,y) - \pi(y) \right| \cdot \max_{y' \in \Omega} |f_J(y')|.$$

$$= 2 \| P^{\tau}(x,\cdot) - \pi(\cdot) \|_{TV} \cdot \| f_J \|_{\infty}$$

Taking the x that maximizes L.H.S.

$$|\lambda_J|^{\tau} \leq 2 \| P^{\tau}(x,\cdot) - \pi(\cdot) \|_{TV}$$

$$|\lambda_J|^{\tau_{mix}} \leq \frac{1}{2} \implies \tau_{mix} \log\left(\frac{1}{|\lambda_J|} \right) \geq \log 2.$$

$$\tau_{mix} \left(\frac{1}{|\lambda_J|} - 1 \right) \geq \tau_{mix} \log \frac{1}{|\lambda_J|} \geq \log 2.$$

$$\tau_{mix}\left[\frac{\lambda(P)}{1-\lambda(P)}\right] \geq \log 2$$

$$\tau_{mix} \geq \lambda(P)^{-1}(1-\lambda(P))\log 2.$$

$$\geq (\lambda(P)^{-1}-1)\log 2$$