[Ei]genvalues of the transition matrix and mixing times:

$M = (P, \Pi, \Omega)$.

- P is a square $|\Omega| \times |\Omega|$ matrix.
- Let $\{f_i\}_{i=1}^{|\Omega|}$, $\{\lambda_i\}_{i=1}^{|\Omega|}$ be its eigenfunctions and eigenvalues

$$P f_i = \lambda_i f_i \qquad \left[\text{Recall} \quad \det(P - \lambda I) = 0\right].$$

- λ_i may be complex.

Properties: [P irreducible, aperiodic]

(i) $|\lambda_i| \leq 1$

$$\|Pf_i\|_\infty = \max_x |Pf_i(x)| = \max_x \left|\sum_y P(x,y)\cdot f_i(y)\right|$$

$$\leq \max_x \sum_y |P(x,y)\cdot f_i(y)|$$

$$\leq \max_y \|f_i(y)\|_\infty = \|f_i\|_\infty$$

$$\|Pf_i\|_\infty = \|\lambda_i f_i\|_\infty \leq \|f_i\|_\infty$$

$$|\lambda_i| \|f_i\|_\infty \leq \|f_i\|_\infty$$

$$|\lambda_i| \leq 1.$$

(ii) P reversible $\Rightarrow$ $\lambda_i \in \mathbb{R}$ $\forall i$ [proof later].

(iii) $P^{lazy} = \dfrac{I + P}{2}$ has positive eigenvalues
reversible.

$$P^{lazy} \cdot f_i = \frac{f_i + \lambda_i \cdot f_i}{2} = \frac{1+\lambda_i}{2} \cdot f_i \qquad \frac{1+\lambda_i}{2} > 0$$

(iv) $P \cdot \vec{1} = \vec{1}$

$$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq -1.$$

Definition (Absolute Spectral GAP)

$$\lambda_* = 1 - \max_{i \geq 2} |\lambda_i|$$

Spectral GAP: $\lambda(P) = 1 - \lambda_*$.

Thm: M reversible and ergodic. Then,

$$(\lambda(P)^{-1} - 1) \log 2 \leq T_{mix} \leq \lambda(P)^{-1} \cdot \log\left(\frac{4}{\pi_{min}}\right)$$

$$\boxed{\pi_{min} = \min_{x \in \Omega} \pi(x)}$$

Example 1 Lazy-SRW on hypercube:

$$\Omega = \{0,1\}^n$$

$$P(x,y) = \begin{cases} 1/2 & x = y \\ 1/2n & |x-y| = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$\pi(x) = \frac{1}{2^n} \quad \forall x \in \Omega.$$

- Easier (for the computation) to think of the walk on $\{-1,1\}^n$, that has exactly the same behavior.

- $\Omega = \{-1,1\}^n$.

For $S \subseteq [n] = \{1,\dots,n\}$, $x \in \Omega$, let:

$$\phi_S(x) = \prod_{x \in S} x_i$$

<u>Claim:</u> $\{\phi_S(x)\}_{S \subseteq [n]}$ are the eigenfunctions of P with eigenvalues $\frac{n-|S|}{n} =: \lambda_S$

$$P \cdot \phi_S(x) = \sum_{y \in \Omega} P(x,y) \cdot \phi_S(y)$$

$$\boxed{x^{[i]}_j = x_j \ \forall j \neq i}$$

$$= \frac{1}{2} \cdot \phi_S(x) + \frac{1}{2n} \sum_{i=1}^{n} \phi_S(x^{[i]})$$

$$= \frac{1}{2} \phi_S(x) + \frac{1}{2n} \left[\sum_{i \in S} \phi_S(x^{[i]}) + \sum_{i \notin S} \phi_S(x^{[i]}) \right]$$

$$= \frac{1}{2} \phi_S(x) + \frac{1}{2n} \left[-|S| \cdot \phi_S(x) + (n-|S|) \cdot \phi_S(x) \right]$$

$$= \phi_S(x) \left[\frac{1}{2} - \frac{|S|}{2n} + \frac{n-S}{2n} \right]$$

$$= \phi_S(x) \left[\frac{2n - 2|S|}{2n} \right] = \phi_S(x) \frac{n-|S|}{n}.$$

Hence, $\lambda_* = \frac{n-1}{n}$, $\lambda(P) = \frac{1}{n}$, and so

$$(1) \quad \Omega(n) \leq \tau_{mix} \leq \log\left[\frac{4}{\pi_{min}}\right] \cdot n = \Theta(n^2) \qquad \boxed{\pi_{min} = \frac{1}{2^n}}$$

<u>Conductance:</u>

$$S = \{ x : x_1 = 1 \}$$

$$\Phi(S) = \frac{|\partial S|}{2 \sum_{x \in S} \deg(x)} = \frac{2^{n-1}}{2 \cdot 2^{n-1} \cdot n} = \frac{1}{2n}.$$

So,

$$\Phi_* \leq \frac{1}{2n}.$$

<u>Indeed</u>: $\Phi_* = \Theta(n)$ [Ex: check this]

Recall $\tau_{mix} \leq \frac{1}{\Phi_*^2} \cdot \log\left(\frac{4}{\pi_{min}}\right) = O(n^3)$

[worse bound than (1)]

<u>Example 2</u>: <u>L-RW on the n-cycle</u>

$$P(x,y) = \begin{cases} 1/2 & y = (x+1) \bmod n \\ 1/2 & y = (x-1) \bmod n. \end{cases}$$

$$\pi(x) = \frac{1}{n} \quad \forall x \in \Omega := \{1, \ldots, n\}.$$

<u>Claim</u> The eigenvalues and eigenfunctions of P are:

$$f_j(x) = \cos\left(\frac{2\pi j x}{n}\right) \qquad \lambda_j = \cos\left(\frac{2\pi j}{n}\right).$$

<u>Proof:</u>

$$Pf_j(x) = \sum_{y\in\Omega} P(x,y)\cdot f_j(y) = P(x,x-1)\cdot f_j(x-1) + P(x,x+1)\, f_j(x+1)$$

$$= \frac{1}{2}\left[\cos\left(\frac{2\pi j(x-1)}{n}\right) + \cos\left(\frac{2\pi j(x+1)}{n}\right)\right]$$

$$= \frac{1}{2}\left[\frac{e^{\frac{2\pi j(x-1)}{n}\cdot i} + e^{-\frac{2\pi j(x-1)}{n}\cdot i}}{2} + \frac{e^{\frac{2\pi j(x+1)}{n}\cdot i} + e^{-\frac{2\pi j(x+1)}{n}\cdot i}}{2}\right]$$

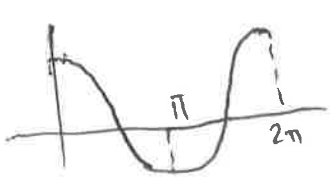
$$= \frac{1}{4}\left[e^{\frac{2\pi j x}{n} i}\left[e^{-\frac{2\pi j}{n} i} + e^{\frac{2\pi j}{n} i}\right] + e^{-\frac{2\pi j}{n}\cdot x}\left[e^{\frac{2\pi j}{n} i} + e^{-\frac{2\pi j}{n} i}\right]\right]$$

$$= \left[\frac{e^{-\frac{2\pi j}{n} i} + e^{\frac{2\pi j}{n}\cdot i}}{2}\right]\left[\frac{e^{\frac{2\pi j x}{n}\cdot i} + e^{-\frac{2\pi j x}{n}\cdot i}}{2}\right]$$

$$= \cos\left(\frac{2\pi j}{n}\right)\cdot\cos\left(\frac{2\pi j x}{n}\right)$$

$$= f_j(x)\cdot\lambda_j.$$

P^{LAZY}: $\quad \lambda_j^L = \frac{1+\lambda_j}{2} = \frac{1}{2}\left(1 + \cos\left(\frac{2\pi j}{n}\right)\right).$



$$\gamma(P^{LAZY}) = 1 - \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right)\right).$$

$$\lambda(P^{LAZY}) = \frac{1 - \cos\left(\frac{2\pi}{n}\right)}{2}$$

Taylor's:

$$1 - \cos\left(\frac{2\pi}{n}\right) = \frac{4\pi^2}{n^2} + O\left(\frac{1}{n^4}\right).$$

$$\tau_{mix} \leq \log\left(\frac{4}{\pi_{min}}\right) O(n^2).$$

$$\boxed{\Omega(n^2) \leq \tau_{mix.} \leq O(n^2 \log n).}$$

Proof upper bound: $\tau_{mix} \leq \log\left(\frac{4}{\pi_{min}}\right) \lambda(P).$

$$\lambda_* = \max_{i \geq 2} |\lambda_i| \qquad \lambda(P) = 1 - \lambda_*.$$

(1) $f, g \in \mathbb{R}^{|\Omega|}$. Let $\langle f, g \rangle_\pi = \sum_x f(x) \cdot g(x) \cdot \pi(x).$

↳ [inner product. (Check)].

$$(\mathbb{R}^{|\Omega|}, \langle \cdot, \cdot \rangle_\pi). = L_2(\pi)$$

⊛ Then, $\{f_j\}_{j=1}^{|\Omega|}$ is an orthonormal basis of $L_2(\pi)$