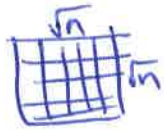
Glauber Dynamics for the Ising Model

Ising Model

$G = (V, E)$ $V =$ 

$\Omega = \{+1, -1\}^{|V|}$

For $\sigma \in \Omega$,

$$\mu(\sigma) = \frac{1}{Z} \exp\Big(\beta \cdot \sum_{u \sim v} \mathbb{1}(\sigma_u = \sigma_v)\Big) \qquad Z = \sum_{\tau \in \Omega} \exp\Big(\beta \sum_{u \sim v} \mathbb{1}(\tau_u = \tau_v)\Big)$$

Glauber dynamics

$X_t \in \Omega$

1) Pick $v \in V$ u.a.r.

2) Update the spin of v with a sample from:

$$\mu(\cdot \mid X_t(V \setminus v)).$$

Let $N_v^+(X_t) =$ # of "+" in neighborhood of v in X_t.

$N_v^-(X_t) =$ # of "−" " "

Then,

$$\mu(v = + \mid X_t(V \setminus v)) = \frac{e^{\beta N_v^+(X_t)}}{e^{\beta N_v^+(X_t)} + e^{\beta N_v^-(X_t)}}.$$

$$\mu(v=- \mid X_\tau(V\setminus v)) = \frac{e^{\beta N_v^-(X_\tau)}}{e^{\beta N_v^-(X_\tau)} + e^{\beta N_v^+(X_\tau)}}$$

Proof:

$$\mu(v=+ \mid X_\tau(V\setminus v)) = \frac{\mu(v=+, X_\tau(V\setminus v))}{\mu(v=+, X_\tau(V\setminus v)) + \mu(v=-, X_\tau(V\setminus v))}$$

$\Rightarrow$ GD dynamics is ergodic and reversible w.r.t. μ.

$\boxed{\sigma_v(u) = \sigma(u)\ \forall u \neq v \text{ and } \sigma_v(v) = -\sigma(v)}$

Typo: sigma_v should be sigma^(v)

$$\mu(\sigma)\cdot P(\sigma,\sigma^{(v)}) = \mu(\sigma^{(v)})\cdot P(\sigma^{(v)},\sigma)$$

$$= \frac{1}{Z}\exp\Big[\beta \sum_{\substack{u\sim w\\ u,w\neq v}} \mathbb{1}(\sigma_u=\sigma_w)\Big]\exp\Big[\beta\cdot N_v^{\sigma(v)}(\sigma)\Big]\cdot\frac{e^{\beta N_v^{-\sigma(v)}(\sigma)}}{e^{\beta N_v^{\sigma(v)}(\sigma)}+e^{\beta N_v^{-\sigma(v)}(\sigma)}}$$

$$= \frac{1}{Z}\exp\Big[\beta \sum_{\substack{u\sim w\\ u,w\neq v}}{}' \mathbb{1}(\sigma_u=\sigma_w)\Big]\exp\Big[\beta\cdot N_v^{-\sigma(v)}(\sigma)\Big]\cdot\frac{\exp[\beta N_v^{\sigma(v)}(\sigma)]}{e^{\beta N_v^{\sigma(v)}(\sigma)}+e^{\beta N_v^{-\sigma(v)}(\sigma)}}$$

$$= \mu(\sigma^{(v)})\cdot P[\sigma^{(v)},\sigma]. \qquad \blacksquare$$

Mixing time: $\beta_c = \ln(1+\sqrt{2})$

(i) If $\beta < \beta_c$, then $T_{mix} = O(n\log n)$

(ii) If $\beta > \beta_c$, then $T_{mix} = e^{\Omega(\sqrt{n})}$.

Intuition: (i) Coupon collecting

(ii) 

Bottleneck btw. mostly "+" and mostly "−" configurations.

⇒ We will prove (i) today.

⇒ In fact we'll prove something slightly weaker.

Thm 1: If $\beta < \beta_c$, then $T_{mix} = O(n \log^3 n)$.

[Proof due to Dyer, Sinclair, Vigoda, Weitz '04].

Lemma 2: Suppose $T_{mix}(\sqrt{n} \times \sqrt{n}) \le g(n)$ where $g : \mathbb{N} \to \mathbb{R}^+$ is an increasing function.

Then, $T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot g((\log n)^2)$.

Proof of Thm 1:

Suppose we know $T_{mix}(\sqrt{n} \times \sqrt{n}) \le e^{cn}$.

Then $T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot e^{c(\log n)^2} \le e^{c_1 (\log n)^2}$

$$T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot e^{c_1 (2 \log \log n)^2} \le e^{c_2 \log n}$$

$$\le n^{c_2}$$

$$T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot (\log n)^{2c_2}$$

$$T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot (\log n)^2 (\log \log n)^{c_3}$$

$$T_{mix}(\sqrt{n} \times \sqrt{n}) \le n \cdot (\log n)^2 (\log \log n)^{c_4}$$

Proof of Lemma 2

Idea: Design coupling s.t. after $T = n \cdot g((\log n)^2)$ steps:

$$\Pr[X_T \neq Y_T] \leq 1/4$$

and the result follows.

Identity coupling:

⇒ Pick same (random) vertex v in both copies.

⇒ Use same random coin to sample from $\mu(\cdot \mid X_t(V \setminus v))$ and $\mu(\cdot \mid Y_t(V \setminus v))$.

- Suppose "+" ≥ "−" (total order of the spins).
- Then $X_t \geq Y_t$ if $X_t(v) \geq Y_t(v)$ $\forall v \in V$. (partial order of the state space).

Key fact: The identity coupling is <u>monotone</u>

That is, if $X_t \geq Y_t$, then

$X_{t+1} \geq Y_{t+1}$.

Proof:
- Suppose v is the vertex choosen to be updated
- Then X_t has at least the same number of "+" in the neighborhood of v as Y_t.

$$\left(N_v^+(X_t) \geq N_v^+(Y_t) \right).$$

Then

$$\mu(v=+ \mid X_t(V\setminus v)) = \frac{e^{\beta N_v^+(X_t)}}{e^{\beta N_v^+(X_t)} + e^{\beta N_v^-(X_t)}}$$

$$= \frac{1}{1 + e^{\beta(N_v^-(X_t) - N_v^+(X_t))}} = \frac{1}{1 + e^{\beta[d - 2N_v^+(X_t)]}}$$

$$\geq \frac{1}{1 + e^{\beta[d - 2N_v^+(Y_t)]}} = \mu(v=+ \mid Y_t(V\setminus v)).$$

So, we need to prove (assuming X_0 = all "+", Y_0 = all "−")

$$\Pr[X_T \neq Y_T] \leq 1/4 \qquad T = n \cdot g((\log n)^2).$$

$$\Pr[X_T \neq Y_T] \leq \sum_{v \in V} \Pr[X_T(v) \neq Y_T(v)] \quad \text{(union bound).}$$

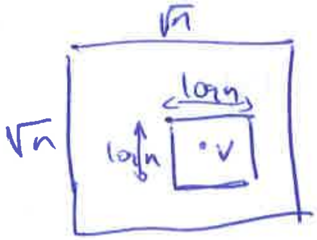
So, enough to show:

$$\Pr[X_T(v) \neq Y_T(v)] \leq \frac{1}{4n}.$$

$\boxed{\text{assuming } X_0 = \text{"+" and } Y_0 = \text{"−"}}$

Let $\{Z_t^+\}$ be an auxiliary copy of the Markov chain. such that:

- $Z_0^+ =$ all "+".
- $\{Z_t^+\}$ does not perform any updates outside of a box centered at v of side length $\log n$.



Similarly define $\{Z_t^-\}$. ($Z_0^- =$ all minus and $\{Z_t^-\}$ does not update in box).

Then, by monotonicity:

$$\Pr[X_t(v) \neq Y_t(v)] \leq \Pr[Z_t^+(v) \neq Z_t^-(v)]$$

Since $Z_t^- \leq Y_t \leq X_t \leq Z_t^+ \quad \forall t \geq 0$

Now, $\Pr[Z_t^+(v) \neq Z_t^-(v)] = \Pr[Z_t^+(v) = +, Z_t^-(v) = -]$

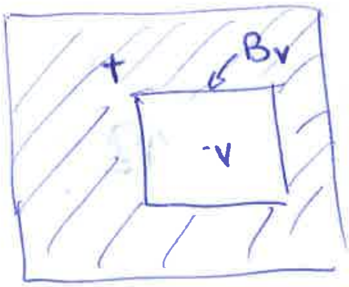
$$= \Pr[Z_t^-(v) = - \mid Z_t^+(v) = +] \cdot \Pr[Z_t^+(v) = +]$$

$$= (1 - \Pr[Z_t^-(v) = + \mid Z_t^+(v) = +]) \Pr[Z_t^+(v) = +]$$

$$= \Pr[Z_t^+(v) = +] - \Pr[Z_t^-(v) = +]$$

Then,

$$\Pr\left[X_t(v) \neq Y_t(v)\right] \le \Pr\left[Z_t^+(v)=+\right] - \Pr\left[Z_t^+(v)=+\right]$$



Idea: after $T = n\,g((\log n)^2)$ steps, how many hit B?

$$\approx \frac{(\log n)^2}{n} \cdot n\, g((\log n)^2)$$

$$= g((\log n)^2)(\log n)^2,$$

which by assumption is much larger than

$$T_{mix}(\log n \times \log n) = g((\log n)^2).$$

⇒ So, the GD is very well-mixed in B_v.

⇒ The stationary measure in B_v is $\mu(\cdot \mid B^c = \text{“}+\text{”})$ for $\{Z_t^+\}$ and $\mu(\cdot \mid B_v^c = \text{“}-\text{”})$ for $\{Z_t^-\}$.

⇒ Then,

$$\Pr\left[Z_T^+(v)=0\right] \approx \mu^*(v=+ \mid B_v^c = \text{“}+\text{”}).$$

$$\Pr\left[Z_T^-(v)=0\right] \approx \mu(v=+ \mid B_v^c = \text{“}-\text{”})$$

⇒ All that we need is then

$$\mu(v=+ \mid B_v^c = \text{“}+\text{”}) \approx \mu(v- \mid B_v^c = \text{“}-\text{”}).$$

$\Rightarrow$ This property holds for $\beta < \beta_c$:

$$(*) \; \left| \mu(v = + \mid B_v^c = \text{"+"}) - \mu(v = + \mid B_v^c = \text{"−"}) \right|$$

$$\leq \exp\left(-\mathrm{dist}(v, \partial B_v)\right) \lesssim \frac{1}{12n}$$

$\rightarrow$ [This property is called Strong Spatial Mixing]

$\Rightarrow$ Consequently, $\Pr[X_T(v) \neq Y_T(v)]$ should be "small". Let's quantify it:

$$\Pr[X_T(v) \neq Y_T(v)] \leq \Pr[Z_T^+(v) = \text{"+"}] - \Pr[Z_T^-(v) = \text{"−"}]$$

$$\leq \left| \Pr[Z_T^+(v) = +] - \mu(v = + \mid B_v^c = +) \right| \quad (A)$$

$$+ \left| \Pr[Z_T^-(v) = +] - \mu(v = + \mid B_v^c = -) \right| \quad (B)$$

$$+ \left| \mu(v = + \mid B_v^c = +) - \mu(v = + \mid B_v^c = -) \right| \quad (C)$$

$\Rightarrow$ By the Strong Spatial Mixing property (C) $\leq \frac{1}{12n}$.

$\Rightarrow$ Note that (A) $\leq \frac{1}{12n}$ also, since after $T = n \cdot g((\log n)^2)$

$$\| Z_T^+ - \mu(\cdot \mid B_v^c = +) \|_{TV} \leq \frac{1}{4(\log n)^2} \leq \frac{1}{12n}$$

Recall that if:

$$\| P^{\tau}(x_0, \cdot) - \mu \|_{TV} \le 1/4,$$

then

$$\| P^{\tau k}(x_0, \cdot) - \mu \|_{TV} \le 1/4^k$$

Putting all these together.

$$A + B + C \le \frac{3}{12n} = \frac{1}{4n}$$

$\blacksquare$