Random matchings:

Given a graph $G=(V,E)$,
let Ω = all matchings of G.

Markov chain on Ω:

From $X_t \in \Omega$,

1. Choose $e \in E$ uniformly at random
2. Let $X' = X_t \oplus e = \begin{cases} X_t \setminus e & \text{if } e \in X_t \\ X_t \cup e & \text{if } e \notin X_t \end{cases}$
3. If $X' \in \Omega$, set $X_{t+1} = X'$ with prob. $1/2$
   otherwise set $X_{t+1} = X_t$.

Ergodic & symmetric $\Rightarrow$ unique stationary is π = uniform(Ω).

Alternative chain:

From $X_t \in \Omega$,

1. Choose $e=(u,v) \in E$ u.a.r.

remove → 2. If $e \in X_t$, then set $X' = X_t \setminus e$.

add → 3. If u & v are unmatched in X_t then $X' = X_t \cup e$.

slide → 4. If u unmatched in X_t & $(v,w) \in X_t$ then set $X' = X_t \cup (u,v) \setminus (v,w)$

If X' is defined then $X' \in \Omega$. → 5. If $X' \in \Omega$ then set $X_{t+1} = X'$ with prob. $1/2$
otherwise set $X_{t+1} = X_t$.

This new chain is also ergodic & symmetric so unique $\pi = \text{uniform}(\Omega)$.

Canonical paths:

For all $I, F \in \Omega$, define a path γ_{IF}

where $\gamma_{IF} = (M_0, M_1, \ldots, M_\ell)$,

$M_0 = I$, $M_\ell = F$

and for all $0 \le i < \ell$, $P(M_i, M_{i+1}) = \frac{1}{2m} > 0$.

For transition $t = M \to M'$ let

$$\rho(t) = \frac{1}{P(M, M')} \sum_{\substack{I, F \in \Omega: \\ \gamma_{IF} \ni t}} 1 = \frac{2m \, |P_{M,M'}|}{|\Omega|}$$

where $P_{M,M'} = \{(I, F) : \gamma_{IF} \ni t\}$

Then $\Phi_* \ge \frac{1}{\rho_*^2}$ where $\rho_* = \max_t \rho(t)$.

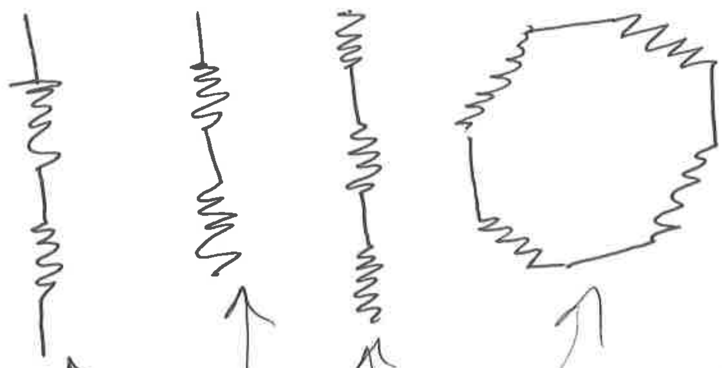
& hence $T_{mix} = O\left(\rho_*^2 \log\left(\frac{1}{\pi_{min}}\right)\right)$

First, fix an arbitrary ordering on vertices $V = \{v_1, v_2, \ldots, v_n\}$

Consider $I \oplus F := (I \setminus F) \cup (F \setminus I)$.

It consists of augmenting paths, deaugmenting paths, alternating paths, & alternating cycles.

For example, $\underline{I}$ & F



Order components by min vertex in each.

Then "unwind" components in order.

For deaugmenting path, remove smaller of 2 ends then do sequence of slides.

For alternating path, remove end in I then do sequence of slides plus insert at end.

For augmenting path, start at smaller of 2 ends then do sequence of slides plus insert at end.

For cycle, remove edge of I incident smallest vtx, then do seq. of slides, plus insert at end.

Consider transition $t = M \to M'$.

Let's suppose t is a slide, $M' = M \setminus (u,v) \cup (v,w)$.

(Similar argument for adds or deletes.)

Want to bound $\mathcal{P}_{M,M'}$

Define $\eta_t : \mathcal{P}_{M,M'} \to \Omega$

Let $E = \eta_t(I,F)$ which we'll set as:

$$E = \Big((I \cup F) \setminus \big(M \cup (u,v) \cup (v,w)\big)\Big) \cup (I \cap F)$$

$$= (I \cap F) \cup (I \oplus F \setminus t)$$

Look at $M \oplus E$. First, $M \cap E = I \cap F$.

Note, $M \oplus E = (I \oplus F) \setminus \big((u,v) \cup (v,w)\big)$

So we know $I \oplus F$, we just need to figure out which edges are in I & which are in F.

Look at the components & order by min vtx.

Say $M \to M'$ (i.e., edges $(u,v),(v,w)$) are in the i^{th} component.

Hence, in components $1 \to i-1$,

M is the same as F

E " I

in components $i+1 \to \ell$

E is the same as F

M is the same as I.

In i^{th} component, the slide $(u,v) \to (v,w)$ tells us the current position of unwinding.

Earlier portion agrees with:

M is the same as F

E is the same as I

& later portion:

E same as F

M same as I.

Problem: if i^{th} component is a cycle then $E \notin \Omega$.

Need to also drop 1^{st} edge of cycle, otherwise E has 2 edges incident to min vtx.

So need to remember if in case 1, 2, 3, or 4.

Hence, $\rho(t) \leq 4|\Omega|$.

Therefore, $T_{mix} = O(m^2 n \log n)$.

What about perfect matchings?

Parameter $\lambda > 0$.

Matching M has weight $w(M) = \lambda^{|M|}$

Sample from distribution $\pi(M) = \frac{w(M)}{Z}$

where $Z = \sum_{M' \in \Omega} w(M')$

<u>Weighted MC:</u> (using Metropolis filter)

From $X_t \in \Omega$,

1. With prob. $\frac{1}{2}$ set $X_{t+1} = X_t$, else:
2. Choose $e \in E$ u.a.r. Let $e = (u, v)$.
3. If u & v unmatched in X_t, set $X' = X_t \cup e$
4. If $e \in X_t$, set $X' = X_t \setminus e$.
5. If u unmatched in X_t & $(v, w) \in X_t$, set $X' = X_t \cup e \setminus (v, w)$.
6. Set $X_{t+1} = X'$ with prob. $\min\left\{1, \frac{w(X')}{w(X_t)}\right\}$

   else $X_{t+1} = X_t$.

Using Metropolis filter $\min\left\{1, \frac{w(new)}{w(old)}\right\}$

has desired stationary distribution.

<u>Canonical paths:</u>

For pairs $I, F \in \Omega$ define path γ_{IF}.

For transition $t = M \to M'$,

congestion $$\rho(t) = \frac{1}{\pi(M)P(M,M')} \sum_{(I,F)\in P_{M,M'}} \pi(I)\pi(F)$$

Note: $\pi(M)P(M,M') = \pi(M')P(M',M)$

So consider $M \to M'$ where $\pi(M') \geq \pi(M)$

& hence $P(M,M') = \frac{1}{2m}$

Then $$\rho(t) = \frac{2m}{|\Omega|\cancel{2m}} \sum_{(I,F)\in P_{M,M'}} \lambda^{|I|+|F|-|M|}$$

In the encoding E + transition M,

how does $|E| + |M|$ compare to $|I| + |F|$?

Lose ≤ 2 edges: initial edge of current component
& current slide

Hence, $\rho(t) \leq 2m\hat{\lambda}^2$ where $\hat{\lambda} = \max\{1, \lambda\}$

& thus $T_{mix} = O(\hat{\lambda}^2 m^2 n \log n)$