For an undirected graph \( G = (V,E) \) where \( n = |V| \) is even, let \( \mathcal{P} \) = set of perfect matchings of \( G \).

Can we compute \( |\mathcal{P}| \) in \( \text{poly}(n) \) time?

No in general. Next class: \( \#P \)-complete for bipartite \( G \).

[Kasteleyn '67]: Poly-time algorithm for planar graphs using the determinant so \( O(n^3) \) time.

[Temperley-Fisher '61]:

Recall, for an \( n \times n \) matrix \( A \) its determinant is defined as:

\[
\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i \in [n]} A(i, \sigma(i))
\]

where \( S_n \) is the set of permutations of \( [n] = \{0, 1, \ldots, n-1\} \) and \( \text{sgn}(\sigma) = (-1)^{N(\sigma)} \) for \( N(\sigma) = \# \) of inversions in \( \sigma \).

It will be useful to consider some equivalent forms of \( \text{sgn}(\sigma) \):

Let \( \sigma = \gamma_1 \cdots \gamma_k \) be its cycle decomposition.

Then, \( \text{sgn}(\sigma) = \prod_{i=1}^{k} \text{sgn}(\gamma_i) \) and \( \text{sgn}(\gamma_i) = (-1)^{1 \mid \gamma_i \mid - 1} \) if \( 1 \mid \gamma_i \mid \) is even

Thus, \( \text{sgn}(\sigma) = (-1)^{\# \text{even cycles in } \sigma} \)

Also, since \( \text{sgn}(\sigma_i) = (-1)^{1 \mid \sigma_i \mid - 1} \) we have:

\[
\text{sgn}(\sigma) = \prod_{i=1}^{n-k} (-1)^{1 \mid \sigma_i \mid - 1} = (-1)^{n-k} \]

\# of cycles in \( \sigma \).
We will orient the edges of undirected $G$ to make a directed graph $\tilde{G}$. We'll do this in such a way that the determinant of the adjacency matrix equals the square of the # of perfect matchings in $G$.

What's an orientation?

For each edge $(i,j) \in E$ we replace it by $ij$ or $ji$.

The original undirected graph is $G=(V,E)$ and the new directed graph is $\tilde{G}=(V,E)$.

Let $A$ be the adjacency matrix of $G$ & we'll use $\tilde{A}$ for the skew-symmetric adjacency matrix of $\tilde{G}$:

$$\tilde{A}(i,j) = \begin{cases} 1 & \text{if } \overrightarrow{ij} \in E \\ -1 & \text{if } \overrightarrow{ji} \in E \\ 0 & \text{if } (i,j) \in E \end{cases}$$

We need an orientation satisfying the following property:

**Definition 1:** For an orientation $\tilde{G}$ of an undirected graph $G$, an even length cycle $C$ in $G$ is oddly oriented if when we traverse $C$ we have an odd # of edges in the opposite direction.

(since $C$ is even it doesn't matter which direction we traverse)
Observation: For a pair of perfect matchings $P, P' \in \mathcal{P}$, $PUP'$ consists of vertex disjoint even length cycles and single edges ($= PNP'$).

Therefore the following definition is well-defined.

Defn. 2: For undirected $G$, orientation $\overrightarrow{G}$ is Pfaffian if $\forall P, P' \in \mathcal{P}$ all cycles of $PUP'$ are oddly oriented.

If we can find a Pfaffian orientation $\overrightarrow{G}$ of $G$, then we can compute $|P|$.

Theorem: For any Pfaffian orientation $\overrightarrow{G}$ of $G$,

$$|P|^2 = \det(\overrightarrow{A}).$$

Proof: First off we can reduce $\#\text{Perfect-Match} \Rightarrow \#\text{Even-cycle-covers}$.

For a directed graph an even cycle cover is a set of vertex disjoint directed cycles, all of even length which cover all of the vertices.
For undirected $G = (V,E)$ let $\overrightarrow{G} = (V,\overrightarrow{E})$ be defined by
for each $(i,j) \in E$, add $\overrightarrow{ij}$ and $\overrightarrow{ji}$ to $\overrightarrow{E}$. Thus
we replace each undirected edge by a pair of antiparallel edges. Let $E$ denote the set of even cycle covers
of $\overrightarrow{G}$.

Lemma: $|P|^2 = |E|$. \\

Proof:

$(\Rightarrow)$: Consider $P, P' \in E$. $P \cup P'$ is a set of disjoint even
length cycles, where $e \in P \cup P'$ will be a cycle of length 2.
For each cycle $C \in P \cup P'$, we can make it a
directed cycle in $\overrightarrow{G}$ in 2 ways. Let's fix
the following way: Assume an arbitrary ordering on
the vertices $V$. Let $v$ be the min $vtx.$ in $C$.
Orient the edge incident to $v$ in $P$ away from $v$
and then follow in that direction around $C$.
This mapping is invertible, so $|P|^2 \leq |E|$. \\

$(\Leftarrow)$: Given $e \in E$, for each $\overrightarrow{C} \in \overrightarrow{E}$, there are 2 ways
to assign the edges to a pair of perfect matchings
$\overrightarrow{P}, \overrightarrow{P'}$. Assign the edge out of $v$ (the min $vtx.$) to $\overrightarrow{P}$
& then alternate $\overrightarrow{P}, \overrightarrow{P'}$ around $\overrightarrow{C}$.
Hence, $|E| \leq |P|^2$. 
Now let's show that: \( \det(G) = |E| \) when \( G \) is Pfaffian.

Consider a permutation \( \sigma = \sigma_1 \cdots \sigma_k \).

Suppose \( \sigma \) contains \( \geq 1 \) cycle of odd-length & let \( \sigma_j \) be the first such one. Let \( V_j \) be the vertices on cycle \( \sigma_j \).

Let \( \sigma' = \sigma_{j-1} \sigma_j^{-1} \sigma_j \cdots \sigma_k \) be the permutation obtained by reversing \( \sigma_j \) & keeping the others as is.

Note, \( \text{sgn}(\sigma) = \text{sgn}(\sigma') = (-1)^{n-k} \).

Since \(|\sigma_j|\) is odd, consider traversing this cycle in direction \( \sigma_j \) vs. direction \( \sigma_j^{-1} \). How many edges of \( G \) are in opposite direction to the traversal?

It's different parity: \( \# \) in opposite direction for \( \sigma_j \) \( \neq \) \( \# \) in opposite direction for \( \sigma_j^{-1} \) mod 2.

Therefore, \( \sum_{i \in V_j} A(i, \sigma(i)) = -\sum_{i \in V_j} A(i, \sigma'(i)) \)

and we have: \( \sum_{i \in [n]} A(i, \sigma(i)) = -\sum_{i \in [n]} A(i, \sigma'(i)) \)

Since the signs are the same: \( \text{sgn}(\sigma) \sum_{i} A(i, \sigma(i)) + \text{sgn}(\sigma') \sum_{i} A(i, \sigma'(i)) = 0 \).

So these terms cancel.
Thus we're left with: \[ \det(A^\rightarrow) = \sum_{s\in \mathfrak{S}_n} \prod_{i\in [n]} A(i, \sigma(i)) \]

where \( E_n \) is the set of permutations consisting of only even-length cycles.

Note: \[ \prod_{i\in [n]} A(i, \sigma(i)) \neq 0 \text{ iff } \forall i \in [n], (i, \sigma(i)) \in E \]

Thus, the only non-zero terms are \( \sigma \) corresponding to an even-cycle cover in \( G \).

Therefore we now have:

\[ \det(A^\rightarrow) = \sum_{\sigma \in E} \prod_{i\in [n]} A(i, \sigma(i)) \]

For \( \sigma \in E \), since \( G \) is Pfaffian for \( \chi \in \mathcal{O} \),

\[ \prod_{i\in V_1} A(i, \sigma(i)) = -1 \text{ since } |V_1| \text{ is even.} \]

But also hence:

\[ \prod_{i\in [n]} A(i, \sigma(i)) = (-1)^{|E_{\text{even}}(\sigma)} \]

But also:

\[ \text{sgn}(\sigma) = (-1)^{|E_{\text{even}}(\sigma)|} = (-1)^{k(\sigma)} \]

Therefore, \( \det(A^\rightarrow) = \sum_{\sigma \in E} (-1)^{k(\sigma)} \).
When can we find a Pfaffian orientation?

Lemma: For a planar $G$, we can construct a Pfaffian orientation $\vec{G}$ in poly-time.

Proof:
First we’ll make an orientation $\vec{G}$ where every face, except possibly the outer face, has an odd # of clockwise edges.
Part of face is on right-side when traversing the edge in forward direction.

Then we’ll prove this orientation is Pfaffian.
We’ll proceed by induction on the # of edges.

Start with a spanning tree of $G$.
Any orientation is OK since the only face is the outer face.

For $|E| > 1|V| - 1$, take an edge $e$ on the outer face. Look at $G\backslash e$ & inductively orient $it$. Adding in $e$ creates $\leq 1$ new face $f$. One of the 2 orientations gives an odd # of clockwise edges on $f$. 
Why is this orientation Pfaffian?

Take a cycle $C$ in $G$.

Look at the induced subgraph in $G$ on $C$ & the vertices inside $C$.

Let $E_{\text{clock}}(C) =$ # clockwise edges on $C$ in $G$.

Let $F =$ # of non-outer faces in this subgraph.

Let $f_1, \ldots, f_F$ be these non-outer faces.

Let $V_{\text{on}}, E_{\text{on}} =$ # of vertices, edges on $C$.

Let $V_{\text{in}}, E_{\text{in}} =$ rest of vertices, edges.

Let $E_{\text{clock}}(f_i) =$ # of clockwise edges on the boundary of $f_i$.

**Euler's formula**: $n - m + f = 2$

Thus, $(V_{\text{on}} + V_{\text{in}}) - (E_{\text{on}} + E_{\text{in}}) + (F + 1) = 2$

$C$ is a cycle so $V_{\text{on}} = E_{\text{on}}$

and we're left with: $V_{\text{in}} - E_{\text{in}} + F = 1 \quad (*)$. 
Our construction of $\vec{G}$ says that for all $i$,

\[ E_{\text{on}}(f_i) \equiv 1 \mod 2. \]

Thus,

\[ \sum_{i=1}^{F} E_{\text{on}}(f_i) \equiv F \mod 2 \tag{***} \]

For each $e$ inside $C$, $e$ is clockwise on exactly 1 non-outer face & $E_{\text{on}}^\text{clock}(C)$ are clockwise on exactly 1 non-outer face.

Thus,

\[ \sum_{i=1}^{F} E_{\text{on}}(f_i) = E_{\text{in}} + E_{\text{on}}^\text{clock}(C). \]

Plugging in (***), we have:

\[ F = E_{\text{in}} + E_{\text{on}}^\text{clock}(C) \mod 2 \]

Plugging in (\*), we then have:

\[ F = V_{\text{in}} + F + E_{\text{on}}^\text{clock}(C) - 1 \mod 2 \]

Therefore: $V_{\text{in}} \neq E_{\text{on}}^\text{clock}(C) \mod 2$. 
Finally, take a pair \( P_i P_j P_k \) and look at a cycle \( CEPUP' \). We know \(|C|\) is even.

Since \( G \) is planar the vertices inside \( C \) are matched with each other so \(|V_{in}|\) is even.

Hence, \( E_{\text{out}}(c) \) is odd.

Therefore, \( G \) is Pfaffian.

\[ \text{Note, [Alon, Robertson, Seymour, Thomas, '97]} \]
give a poly-time alg. to decide for a bipartite \( G \), if there exists a Pfaffian orientation.