For an undirected graph $G=(V,E)$ where $n=|V|$ is even,
let $\mathcal{P}$ = set of perfect matchings of G.

Can we compute $|\mathcal{P}|$ in poly(n) time?

No in general. Next class: #P-complete for bipartite G.

[Kasteleyn '61]: Poly-time algorithm for planar graphs,
using the determinant so $O(n^3)$ time.

[Temperley-Fisher '61] did $\mathbb{Z}^2$.

Recall, for a $n \times n$ matrix A its determinant is defined as:

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in [n]} A(i, \sigma(i))$$

where S_n is the set of permutations of $[n] = \{0, 1, \ldots, n-1\}$

and $\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$ for $N(\sigma) = $ # of inversions in σ.

$\{i,j : i<j, \sigma(i) > \sigma(j)\}$

It will be useful to consider some equivalent forms of $\operatorname{sgn}(\sigma)$:

Let $\sigma = \gamma_1 \cdots \gamma_k$ be its cycle decomposition.

Then, $\operatorname{sgn}(\sigma) = \prod_{i=1}^{k} \operatorname{sgn}(\gamma_i)$ and $\operatorname{sgn}(\gamma_i) = \begin{cases} -1 & \text{if } |\gamma_i| \text{ is even} \\ 1 & \text{if } |\gamma_i| \text{ is odd} \end{cases}$

Thus, $\operatorname{sgn}(\sigma) = (-1)^{\#\{\text{even cycles in } \sigma\}}$

Also, since $\operatorname{sgn}(\sigma_i) = (-1)^{|\gamma_i|-1}$ we have:

$$\operatorname{sgn}(\sigma) = \prod_i (-1)^{|\gamma_i|-1} = (-1)^{n-k}$$

k ← # of cycles in σ.

We will orient the edges of undirected G to make a directed graph $\vec{G}$. We'll do this in such a way that the determinant of the adjacency matrix equals the square of the # of perfect matchings in G.

What's an orientation?

For each edge $(i,j) \in E$ we replace it by $\vec{ij}$ or $\vec{ji}$.

The original undirected graph is $G=(V,E)$ and the new directed graph is $\vec{G}=(V,\vec{E})$.

Let A be the adjacency matrix of G & we'll use $\vec{A}$ for the skew-symmetric adjacency matrix of $\vec{G}$:

$$\vec{A}(i,j) = \begin{cases} 1 & \text{if } \vec{ij} \in \vec{E} \\ -1 & \text{if } \vec{ji} \in \vec{E} \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

We need an orientation satisfying the following property:

<u>Definition 1:</u> For an orientation $\vec{G}$ of an undirected graph G, an even length cycle C in G is oddly oriented if when we traverse C we have an odd # of edges in the opposite direction.

(Since C is even it doesn't matter which direction we traverse)

<u>Observation</u>: For a pair of perfect matchings $P, P' \in \mathcal{P}$,

$P \cup P'$ consists of vertex disjoint even length cycles and single edges $(= P \cap P')$.

Therefore the following definition is well-defined.

<u>Def'n. 2</u>: For undirected G, orientation $\vec{G}$ is <u>Pfaffian</u> if $\forall P, P' \in \mathcal{P}$ all cycles of $P \cup P'$ are oddly oriented.

If we can find a Pfaffian orientation $\vec{G}$ of G, then we can compute $|\mathcal{P}|$.

<u>Theorem</u>: For any Pfaffian orientation $\vec{G}$ of G,

$$|\mathcal{P}|^2 = \det(\vec{A}).$$

<u>Proof</u>:

First off we can reduce #Perfect-Match $\to$ #Even-cycle-covers.

For a directed graph an even cycle cover is a set of vertex disjoint directed cycles, all of even length, which cover all of the vertices.

For undirected $G=(V,E)$ let $\overleftrightarrow{G}=(V,\overleftrightarrow{E})$ be defined by for each $(i,j)\in E$, add $\overrightarrow{ij}$ and $\overrightarrow{ji}$ to $\overleftrightarrow{E}$. Thus we replace each undirected edge by a pair of antiparallel edges. Let $\mathcal{E}$ denote the set of even cycle covers of $\overleftrightarrow{G}$.

<u>Lemma</u>: $|\mathcal{P}|^2 = |\mathcal{E}|$.

<u>Proof</u>:

($\Rightarrow$): Consider $P, P' \in \mathcal{P}$. $P \cup P'$ is a set of disjoint even length cycles, where $e \in P \cap P'$ will be a cycle of length 2.

For each cycle $C \in P \cup P'$ we can make it a directed cycle in $\overleftrightarrow{G}$ in 2 ways. Let's fix the following way. Assume an arbitrary ordering on the vertices V. Let v be the min vtx. in C. Orient the edge incident to v in P away from v and then follow in that direction around C.

This mapping is ~~finder~~ invertible, so: $|\mathcal{P}|^2 \leq |\mathcal{E}|$.

($\Leftarrow$): Given $\sigma \in \mathcal{E}$, for each directed cycle $\overleftrightarrow{C} \in \sigma$, there are 2 ways to assign the edges to a pair of perfect matchings P, P'. Assign the edge out of v (the min vtx.) to P & then alternate P, P' around $\overleftrightarrow{C}$.

Hence, $|\mathcal{E}| \leq |\mathcal{P}|^2$. ∎

Now let's show that: $\det(\vec{A}) = |\mathcal{E}|$ when $\vec{G}$ is Pfaffian.

Consider a permutation $\sigma = \gamma_1 \cdots \gamma_k$.

Suppose σ contains ≥ 1 cycle of odd-length & let γ_j be the first such one. Let V_j be the vertices on cycle γ_j.

Let $\sigma' = \gamma_1 \cdots \gamma_{j-1} \gamma_j^{-1} \gamma_{j+1} \cdots \gamma_k$ be the permutation obtained by reversing γ_j & keeping the others as is.

Note, $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma') = (-1)^{n-k}$.

Since $|\gamma_j|$ is odd, consider traversing this cycle in direction γ_j vs. direction γ_j^{-1}. How many edges of $\vec{G}$ are in opposite direction to the traversal?

It's different parity: $\#$ in opposite direction for $\gamma_j \not\equiv \#$ in opp. dir. for γ_j^{-1} mod 2.

Therefore,
$$\prod_{i \in V_j} \vec{A}(i, \sigma(i)) = -\prod_{i \in V_j} \vec{A}(i, \sigma'(i))$$

and we have:
$$\prod_{i \in [n]} \vec{A}(i, \sigma(i)) = -\prod_{i \in [n]} \vec{A}(i, \sigma'(i))$$

Since the signs are the same:
$$\operatorname{sgn}(\sigma) \prod_i \vec{A}(i, \sigma(i)) + \operatorname{sgn}(\sigma') \prod_i \vec{A}(i, \sigma'(i)) = 0.$$

So these terms cancel.

Thus we're left with: $\mathrm{Det}(\vec{A}) = \sum_{\sigma \in E_n} \overset{\mathrm{Sgn}(\sigma)}{\checkmark} \prod_{i \in [n]} \vec{A}(i, \sigma(i))$

where E_n is the set of permutations consisting of only even-length cycles.

Note: $\prod_{i \in [n]} A(i, \sigma(i)) \neq 0$ iff $\forall i, (i, \sigma(i)) \in E$

Thus, the only non-zero terms are σ corresponding to an even-cycle cover in $\overleftrightarrow{G}$.

Therefore we now have:

$$\mathrm{Det}(\vec{A}) = \sum_{\sigma \in \mathcal{E}} \mathrm{sgn}(\sigma) \prod_i \vec{A}(i, \sigma(i)).$$

For ~~$\sigma \in E_n$~~ $\sigma \in \mathcal{E}$, since $\vec{G}$ is Pfaffian for $\gamma_j \in \sigma$,

$$\prod_{i \in V_j} \vec{A}(i, \sigma(i)) = -1 \quad \text{since } |\gamma_j| \text{ is even.}$$

~~but also~~ hence: $\prod_{i \in [n]} \vec{A}(i, \sigma(i)) = (-1)^{k(\sigma)}$ ← #cycles in σ.

but also: $\mathrm{sgn}(\sigma) = (-1)^{\#\text{even cycles in } \sigma} = (-1)^{k(\sigma)}$ since $\sigma \in E_n$.

Therefore,

$$\mathrm{Det}(\vec{A}) = \sum_{\sigma \in \mathcal{E}} \left((-1)^{k(\sigma)}\right)^2 = |\mathcal{E}|.$$

∎

When can we find a Pfaffian orientation?

Lemma: For a planar G, we can construct a Pfaffian orientation $\vec{G}$ in poly-time.

Proof:

First we'll make an orientation $\vec{G}$ where every face, except possibly the outer face, has an odd # of clockwise edges.

Part of face is on right-side when traversing the edge in forward direction.

Then we'll prove this orientation is Pfaffian.

We'll proceed by induction on the # of edges.

Start with a spanning tree of G.

Any orientation is OK since the only face is the outer face.

For $|E| > |V| - 1$, take an edge e on the outer face. Look at $G \setminus e$ & inductively orient it. Adding in e creates ≤ 1 new face f. One of the 2 orientations of e gives an odd # of clockwise edges on f.

Why is this orientation $\vec{G}$ Pfaffian?

Take a cycle C in G.

Look at the induced subgraph in $\vec{G}$ on C & the vertices inside C.

Let $E_{ON}^{CLOCK}(C) = \#$ clockwise edges on C in $\vec{G}$.

Let $F = \#$ of non-outer faces in this subgraph.

Let $f_1, \ldots, f_F$ be these non-outer faces.

Let $V_{ON}, E_{ON} = \#$ of vertices, edges on C.

Let $V_{IN}, E_{IN} =$ rest of vertices, edges.

Let $E_{ON}^{CLOCK}(f_i) = \#$ of clockwise edges on the boundary of f_i

<u>Euler's formula:</u> $n - m + f = 2$

Thus, $(V_{ON} + V_{IN}) - (E_{ON} + E_{IN}) + (F+1) = 2$

C is a cycle so $V_{ON} = E_{ON}$

and we're left with: $V_{IN} - E_{IN} + F = 1$ $(*)$.

Our construction of $\vec{G}$ says that for all i,

$$E_{ON}^{CLOCK}(f_i) \equiv 1 \mod 2.$$

Thus,

$$\sum_{i=1}^{F} E_{ON}^{CLOCK}(f_i) \equiv F \mod 2 \quad (**)$$

For each e inside C, e is clockwise on exactly 1 non-outer face & $E_{ON}^{CLOCK}(C)$ are clockwise on exactly 1 non-outer face.

Thus,

$$\sum_{i=1}^{F} E_{ON}^{CLOCK}(f_i) = E_{IN} + E_{ON}^{CLOCK}(C).$$

Plugging in $(**)$ we have:

$$F \equiv E_{IN} + E_{ON}^{CLOCK}(C) \mod 2$$

Plugging in $(*)$ we then have:

$$\cancel{F} \equiv V_{IN} + \cancel{F} + E_{ON}^{CLOCK}(C) - 1 \mod 2$$

Therefore: $V_{IN} \not\equiv E_{ON}^{CLOCK}(C) \mod 2.$

Finally, take a pair $P, P' \in \mathcal{P}$ & look at a cycle $C \in P \cup P'$. We know $|C|$ is even.

Since G is planar the vertices inside C are matched with each other so $|V_{IN}|$ is even.

Hence, $E_{ON}^{CLOCK}(C)$ is odd.

Therefore, $\vec{G}$ is Pfaffian.

$\blacksquare$

Note, [~~Mc~~ McCuaig, Robertson, Seymour, Thomas, '97] give a poly-time alg. to decide for a bipartite G, if there exists a Pfaffian orientation.