#P is the counting analog of NP

NP = Decision problems YES/NO: Does there exist a witness?

#P = Counting problem: # of witnesses.

NP-predicate: $\chi : \Sigma^* \times \Sigma^* \to \{0,1\}$ s.t.

a) $\chi(I,w)$ can be computed in time poly$(|I|)$,

b) $\exists$ poly $p(x)$ s.t.

if $\chi(I,w)=1$ then $|w| \leq p(|I|)$.

NP-problem: Does $\exists w$ s.t. $\chi(I,w)=1$?

#P-problem: Compute $f_\chi(I) = |\{w : \chi(I,w)=1\}|$

$f \in$ #P if $\exists$ NP-predicate χ for f.

So a #P-predicate is $f : \Sigma^* \to N = \{0,1,2,\dots\}$

A reduction $\chi \leq \eta$ defined by ϕ is <u>Parsimonious</u> if it preserves the # of solutions, i.e.,

$$f_\chi(I) = f_\eta(\phi(I)).$$

Standard proof that SAT is NP-complete is parsimonious, and hence #SAT is #P-complete.

Similarly, #SAT $\leq$ #3SAT $\leq$ #Exact-3-Cover

Using parsimonious NP-completeness reductions.

#Exact-3-Cover:

Input: Set $X = \{1, \ldots, n\}$ & collection $Y \subseteq \binom{X}{3}$

Output: # of $Z \subseteq Y$ s.t. each $i \in X$ is covered exactly once by Z.

Let A be a $n \times n$ matrix.

Its permanent is defined as:

$$\mathrm{Per}(A) = \sum_{\sigma \in S_n} \prod_{i \in [n]} A(i, \sigma(i))$$

For 0-1 matrix A, let the n rows correspond to $R = \{r_1, \ldots, r_n\}$ and the n columns to $C = \{c_1, \ldots, c_n\}$.

Include edge (r_i, c_j) iff $A(i,j) = 1$

Then, $\mathrm{Per}(A) = $ # of perfect matchings in the bipartite graph $G = (R \cup C, E)$.

#W-BI-MATCH

INPUT: bipartite G with integer edge weights.

OUTPUT: $\sum_{M \in \Omega} w(M) = \sum_{M \in \Omega} \prod_{e \in M} w(e) =$ total weight of matchings of G.

where Ω = all matchings of G (of any size).

Note, #PERM = #W-BI-MATCH

#(0,1)-PERM = # BI-PER-MATCH

#(0,1)-PERM:

INPUT: bipartite G

OUTPUT: $|\mathcal{P}|$ = # of perfect matchings of G.

#(0,1)-ℓ-PERM:

INPUT: Matrix A with $\leq \ell$ dif't. values, or bipartite G with edge weights where $\leq \ell$ values for edge weights $\neq 1$.

OUTPUT: $\sum_{P \in \mathcal{P}} w(P)$ where $\mathcal{P}$ = set of perfect matchings.

Even though BI-PER-MATCH $\in$ P

<u>Theorem:</u> [Valiant '79] #(0,1)-PERM is #P-complete.

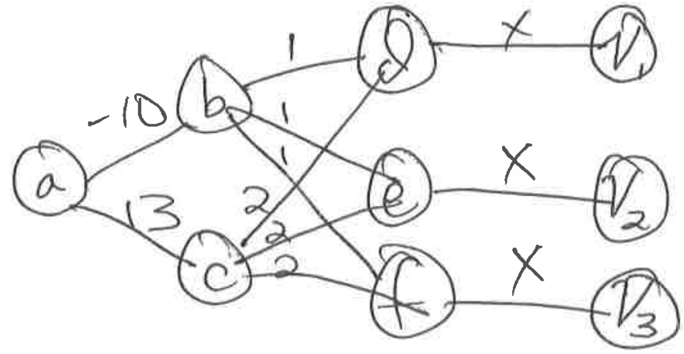
<u>Proof:</u> We'll show #Exact-3-Cover $\leq$ #(0,1)-PERM.
To do this we'll show:

1. #Exact-3-Cover $\leq$ #W-BI-MATCH
2. #W-BI-MATCH $\leq$ #PERM
3. #(0,1)-ℓ-PERM $\leq$ #(0,1)-(ℓ-1)-PERM.

and the theorem then follows.

Proof of (i): #Exact-3-Cover $\leq$ #W-BI-MATCH

Gadget H:



Claim: $w(\Omega(H)) =$ total weight of all matchings of H $= 4(1+x^3)$

Proof:

x^3 term: have to use $(d,v_1),(e,v_2),(f,v_3)$.

then: $x^3 - 10x^3 + 13x^3 = 4x^3$

x^2 term: WOLOG, include $(d,v_1),(e,v_2)$.

then: $x^2 + x^2 + 2x^2 - 10x^2 - 20x^2 + 13x^2 + 13x^2 = 0x^2$

x term: $\ldots = 0x$

constant term: $1 - 10 + 13 \underset{\times 3}{-20} + \underset{\times 3}{13} + \underset{\times 3}{2} + \underset{\times 3}{1} + \underset{\times 6}{2}$

$= -70 + 14 + 39 + 21 = 4$

$\blacksquare$

Consider input $I=(X,Y)$ to #Exact-3-Cover.

Let $S=$ # of solutions.

We'll define a graph $G=G_I$.

For each $i \in X$, add vertices v_i & w_i

and edge (v_i, w_i) with weight -1.

For $A=\{i,j,k\} \in Y$

add $H_A =$ copy of H

identify v_1 with v_i

v_2 with v_j

v_3 with v_k.

Claim: For $x=1$, $w(\Omega(G)) = 4^{|Y|} S$.

Thus given an oracle for #W-BI-MATCH we

can take input I for #EXACT-3-Cover,

with $x=1$ construct G, solve #W-BI-MATCH on G,

divide the solution by 4^m where $m=|Y|$ & we

get S.

Proof of claim:

Let $\mathcal{M}$ = all matchings in G.
Let $\mathcal{M}' \subset \mathcal{M}$ = all matchings which include all v_i's & none of the w_i's.

Note: $w(\mathcal{M} \setminus \mathcal{M}') = 0$.

why? For $M \in \mathcal{M} \setminus \mathcal{M}'$, ~~&~~ let v_i be the smallest i s.t. v_i is not covered in M, or w_i is ~~not~~ covered in M.

~~let $M' = M \cup (v_i, w_i)$.~~

let $M' = M \oplus (v_i, w_i)$.

So if w_i is covered then $(v_i, w_i) \in M$
So $M' = M \setminus (v_i, w_i)$
& if v_i is not covered then w_i is not covered as well so $M' = M \cup (v_i, w_i)$.

Notice that $w(M) = -w(M')$ because (v_i, w_i) has weight -1.

This mapping is bijective so we have $w(\mathcal{M} \setminus \mathcal{M}') = 0$.

For each $M \in \mathcal{M}'$, every v_i is matched & no w_i is matched so no (v_i, w_i) edges are in M.

Hence the v_i are matched using edges from the gadgets H_A's.

For each H_A, We claim that all 3 edges of weight x are used or none of these 3 are used.

Why?

~~Let B be the subgraph of G induced by the v_i & their neighbors.~~

~~Take a matching of~~

For a particular gadget H_A, if we have 1 or 2 edges incident v_1, v_2, v_3 in M then the total weight of matchings for the rest of H_A is 0 since the x & x^2 terms in $w(\Omega(H))$ are 0.

Hence we get weight 4 if 0 or 3 of these are used for each H_A.

It multiplies over the H_A so

$$w(\Omega(G)) = 4^m |S| \text{ where } m = |Y|.$$

$\blacksquare$