Theorem: (0-1)-Permanent is #P-complete

Proof:

$$\#\text{Exact-3-Cover} \underset{1}{\le} \#\text{W-BI-MATCH} \underset{2}{\le} \#\text{PERM}$$

$$\#(0,1)\text{-}\ell\text{-Perm} \le \#(0,1)\text{-}(\ell-1)\text{-Perm}$$

Input: Set $X = \{x_1, \dots, x_n\}$ & collection $Y \subseteq \binom{X}{3}$

Output: # of $Z \subseteq Y$ s.t. each $i \in X$ is covered exactly once.

We showed 1 last class.

#W-BI-MATCH:

input: bipartite G with edge weights

output: $\sum_{M \in \Omega} w(M) = \sum_{M \in \Omega} \prod_{e \in M} w(e)$

where Ω = all matchings of G.

#Perm:

input: bipartite G with edge weights

output: $\sum_{P \in \mathcal{P}} w(P)$ where $\mathcal{P}$ = all perfect matchings of G.

Let's prove 2:

For integer $k \geq 0$, let

$$\#\text{W-}k\text{-BI-MATCH} = \sum_{\substack{M \in \Omega: \\ |M| = k}} w(M) = \text{total weight of all matchings of size } k.$$

We'll show: $\#\text{W-}k\text{-BI-MATCH} \leq \#\text{PERM}$

Then summing over k we get: $\#\text{W-BI-MATCH} \leq \#\text{PERM}$.

Take bipartite $G = (L \cup R, E)$ where $|L| = \ell$, $|R| = r$.

Let k be integer where $0 \leq k \leq \min\{\ell, r\}$.

Form G': Take G,

- add $v_1, \ldots, v_{r-k}$ to L connected to all of R
- add $w_1, \ldots, w_{\ell-k}$ to R connected to all of L.

Each $M \in \Omega(G)$ of size $|M| = k$ corresponds to $(\ell-k)!(r-k)!$ perfect matchings of G'.

Hence, $\#\text{PM of } G' = (\#\text{W-}k\text{-BI-MATCH}) \times (\ell-k)!(r-k)!$

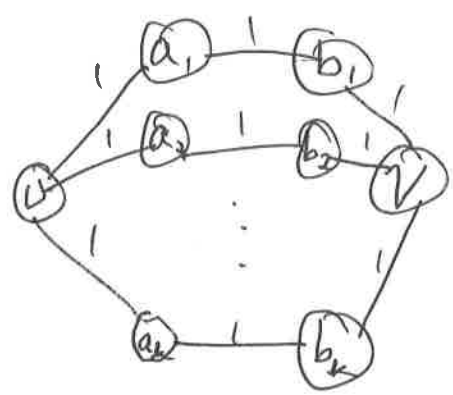
Proof of 3:

$\#(0,1)$-d-Perm $\le$ $\#(0,1)$-$(d-1)$-Perm

input: bipartite G with edge weights where $\le d$ Values on edges that are $\neq 1$.

output: $\sum_{P \in \mathcal{P}} w(P)$.

Gadget F_k:



F_k has k perfect matchings

& 1 matching in $N(u,v)$ = Perfect matching in $G \setminus \{u,v\}$
= cover all vertices except u & v.

Take matrix A corresponding to input G.

We want $per(A)$.

Choose value α we want to eliminate, $\alpha \notin \{0,1\}$

Replace all α by a variable x then

$Per(A)$ = Polynomial $p(x)$ of deg. $\le n$ in x.

Instead: in G replace each edge (u,v) of weight α by a gadget F_k, for a parameter k.

Then: $Per(A) = p(k) =$ polynomial in k.

\# of non-1 entries went $\downarrow$ by 1.

To evaluate $p(\alpha)$,

evaluate $p(k)$ at $k \in \{0, 1, \dots, n\}$ (just change size of gadget)

this yields $p(k)$ at $n+1$ different points.

Since Degree of $p(k) \leq n$ then the value of $p(k)$ at $n+1$ points defines $p(k)$, i.e., we can get the coefficients of $p(k)$ by Gaussian elimination. So interpolate to get $p(k)$ at $k = \alpha$.

$\blacksquare$

Can't hope for efficient exact counting algorithm for many counting problems, such as perfect matchings or all matchings.

Instead: aim for approximation algorithm.

FPRAS: fully-polynomial randomized approximation scheme.

For #(0,1)-Perm:

Given input $G=(V,E)$ & error parameter $\epsilon > 0$, an FPRAS produces OUT s.t.

$$\Pr\left((1-\epsilon)|P| \le OUT \le (1+\epsilon)|P|\right) \ge \frac{3}{4}$$

in time $\mathrm{poly}(n, 1/\epsilon)$.

How to boost success probability?

Given $\delta > 0$.

Want to succeed with prob. $\ge 1-\delta$.

Run FPRAS $k = 100 \ln\left(\frac{2}{\delta}\right)$ times & get outputs $Y_1, \dots, Y_k$.

Let $Z = \text{median}(Y_1, \dots, Y_k)$.

Output Z.

so i^{th} trial was accurate

<u>Analysis:</u> Let $X_i = \begin{cases} 1 & \text{if } Y_i \in (1 \pm \epsilon)|P| \\ 0 & \text{o/w} \end{cases}$

& $X = \sum_{i=1}^{k} X_i$

Note $E[X] \geq \frac{3}{4}k$.

<u>Chernoff's bound:</u>

Let $X_1, \dots, X_m$ be iid $\{0,1\}$ r.v.s where $p = E[X_i]$.

& $X = \sum_{i=1}^{m} X_i$. Let $\mu = E[X] = mp$.

$$\Pr\left(|X - \mu| > \epsilon\mu \right) \leq 2e^{-\epsilon^2 \mu/3}$$

$$\Pr\left(Z \notin (1\pm\epsilon)|\mathcal{P}|\right) \le \Pr\left(X < \frac{k}{2}\right)$$

if $\ge \frac{1}{2}$ of the trials are "good" then the median is good.

$$\le \Pr\left(|X - E[X]| > \frac{k}{4}\right)$$

$$\le 2e^{-k/96}$$

$$\le \delta.$$

Hence to get error prob $\le \delta$, takes $O(\log(1/\delta))$ time.

Approximate sampler:

FPAUS: fully-poly almost uniform sampler

Given $G=(V,E)$, ~~EPA~~ & $\delta > 0$,

FPAUS generates $P \in \mathcal{P}$ from distribution π on $\mathcal{P}$ where $d_{TV}(\mu, \pi) \le \delta$

in time $\text{poly}(n, \log(1/\delta))$

where $\mu = \text{uniform}(\mathcal{P})$.

for μ & π on Ω,

$$d_{TV}(\mu,\pi) = \frac{1}{2}\sum_{x\in\Omega} |\mu(x)-\pi(x)|$$

$$= \max_{S\subseteq\Omega} \mu(S)-\pi(S)$$

$$\begin{tikzcd}
\text{FPRAS} \arrow[r, Leftrightarrow] & \text{FPAUS} \\
\text{Exact counter} \arrow[r, Rightarrow] & \text{Exact-sampler} \arrow[ul]
\end{tikzcd}$$

(for self-reducible problems [JVV])