For an undirected $G=(V,E)$,

let $\Omega(G) =$ spanning trees of G

$= \{S \subset E : S \text{ is acyclic \& connected}\}$

thus $|S| = |V|-1$

Goal: sample uniformly from Ω.

For undirected G, define $\overleftrightarrow{G}$ to be the directed graph where: for $(v,w) \in E$, add $\overrightarrow{vw}$ & $\overrightarrow{wv}$ to $\overleftrightarrow{E}$

For directed $\overrightarrow{G}$, for vertex $r \in V$,

arborescence rooted at $r \in V$:

$S \subset E$ s.t. $|S| = |V|-1$

& every $v \neq r$ has exactly 1 edge directed away from v.

(i.e., its a directed tree rooted at r, pointing ~~away from~~ towards r)

Let $\Omega_r(\overleftrightarrow{G}) =$ set of arborensences rooted at r.

& $\Omega(\overleftrightarrow{G}) = \bigcup_{r \in V} \Omega_r(\overleftrightarrow{G}) =$ all arborescences.

<u>Observation</u>: $|\Omega(G)| = |\Omega_r(\overleftrightarrow{G})|$

\# spanning trees of G = \# arborescences of $\overleftrightarrow{G}$ rooted at r.

<u>Proof</u>:

$\Rightarrow$ for $S \in \Omega(G)$, form $S' \in \Omega_r(\overleftrightarrow{G})$ by pointing all edges ~~away from~~ towards r.

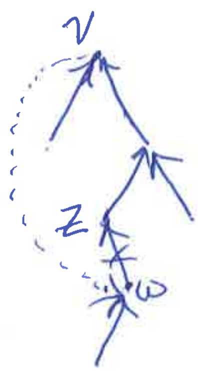
$\Leftarrow$ for $S' \in \Omega_r(\overleftrightarrow{G})$, drop orientations & we have $S \in \Omega(G)$.

∎

<u>MC1 on $\Omega(\overleftrightarrow{G})$</u>: (on all arboresences)

From $X_t \in \Omega(\overleftrightarrow{G})$ with root v.

1. Choose a random (out) neighbor w of v (i.e., $\overrightarrow{vw} \in \overleftrightarrow{E}$)



2. Let $\overrightarrow{wz}$ be the unique edge away from w in X_t
3. Let $X' = X_t \cup \overrightarrow{vw} \setminus \overrightarrow{wz}$.

   Note $X' \in \Omega(\overleftrightarrow{G})$ with root w
4. Let $X_{t+1} = \begin{cases} X' & \text{with prob. } 1/2 \\ X_t & \text{otherwise.} \end{cases}$

If G is d-regular then MC1 is symmetric & thus π is uniform over $\Omega(\overleftrightarrow{G})$.

Fix root $r \in V$.

Here is a new MC call it MC_r on $\Omega_r(\overleftrightarrow{G})$.

From $Y_t \in \Omega_r(\overleftrightarrow{G})$:

1. Run MC1 with $X_0 = Y_t$

   Stop at first time t' where $X_{t'} \in \Omega_r(\overleftrightarrow{G})$

   (i.e., 1st time when root returns to r.)

2. Set $Y_{t+1} = X_{t'}$

Note: this MC_r is symmetric & has uniform over $\Omega_r(\overleftrightarrow{G})$ stationary distribution.

What's the mixing time?

Consider lazy random walk on G.

From $v \in V$, with prob. $1/2$ stay at v,

& with prob. $1/2$ move to a random neighbor.

Cover time = # of steps until random walk visits every vertex at least once from worst starting state.

$\parallel$

T_{cover}

__Lemma:__ $\cancel{E[T_{mix}] \leq}$

$$T_{mix} \leq 4 E[T_{cover}]$$

& for all G,

$$n \log n \leq E[T_{cover}] \leq O(nm).$$

Let's prove the mixing time.

Coupling: For a pair $Y_t, Y_t' \in \Omega_r(\overleftrightarrow{G})$

we'll define a coupling $(Y_t, Y_t') \to (Y_{t+1}, Y_{t+1}')$

where:

for all $Y_0, Y_0' \in \Omega_r(\overleftrightarrow{G})$

$$\Pr(Y_T \neq Y_T') \leq 1/4$$

$$\text{for } T = 4E[T_{cover}].$$

For Y_t, Y_t', let

$$H(Y_t, Y_t') = |Y_t \setminus Y_t'|$$

$= \#$ edges in Y_t & not in Y_t'

thus if $H(Y_t, Y_t') = 0$ then $Y_t = Y_t'$.

What's coupling of $(Y_t, Y'_t) \to (Y_{t+1}, Y'_{t+1})$?

$Y_t \to Y_{t+1}$ uses "excursion"

$$X_0 \to X_1 \to X_2 \to \cdots \to X_t.$$

root: $r \to v_1 \to v_2 \to \cdots \to r$

chosen edge: $\underbrace{(r, v_1), (v_1, v_2), \ldots, (v_{t-1}, r)}$

for $Y'_t \to Y'_{t+1}$ use same sequence of edges)

Then they share these edges, namely,

for v on this excursion,

let i be the last time.

Then, $\overrightarrow{v_i v_{i+1}} \in Y_{t+1}$ & $\overrightarrow{v_i v_{i+1}} \in Y'_{t+1}$.

Once every vertex is visited at least once on an excursion then $Y_t = Y'_t$.

The excursions are just a lazy random walk on G.

The time to visit every vertex at least once is the cover time.

Let $T = 4E[T_{cover}]$.

Then, $\Pr(Y_T \neq Y'_T) \leq \Pr(T_{cover} > T)$

$$= \Pr(T_{cover} > 4E[T_{cover}])$$

$$\leq 1/4.$$

$\boxtimes$