Volume Computation

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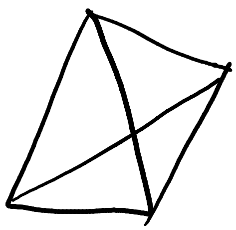
Basic Problem: Find the volume of a given set in $\mathbb{R}^n$.

classical, ancient.

e.g. pyramids (Egypt), Wine barrels (Greece)

Some shapes have nice formulas

e.g. Simplex $= \text{conv}\{x_0, \dots x_n\}$



$$\text{vol} = \frac{1}{n!}\left|\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_n \end{pmatrix}\right|$$

$$\text{Ball}(r) = c_n \cdot r^n$$

$$\text{Ellipsoid} = \{x : x^T A^{-1} x \le 1\}$$

$$\text{vol} = \sqrt{\det(A)} \cdot \text{vol}(B_n).$$

Parallelopiped

$\vdots$

Not a long list.

[illegible]

For a polytope $P = \{x : Ax \geq b\}$

A, b rational. How to compute the volume?

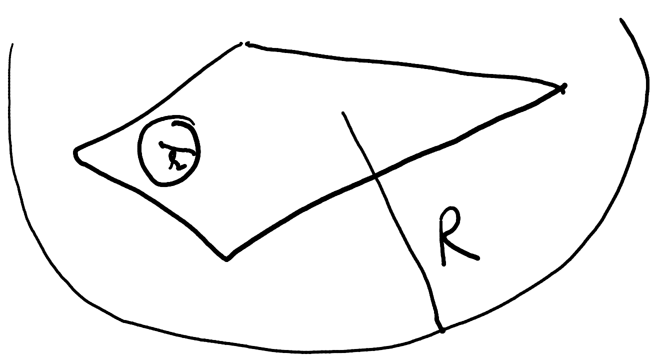
[Dyer-Frieze] Thm. #P-hard.

We can hope to approximate it.

But lets generalize first.

<u>Input</u>: body K given by a "well-guaranteed" membership oracle

r, R: $x_0 + rB_n \subseteq K \subseteq RB_n$



$x \in K$? $\nearrow$ YES $\searrow$ NO

$\varepsilon > 0$.

<u>Output</u>: V s.t. $(1-\varepsilon)\,\mathrm{Vol}(K) \leq V \leq (1+\varepsilon)\,\mathrm{Vol}(K)$

Even more general:

<u>Input</u>: $f : \mathbb{R}^n \to \mathbb{R}_+$ nonnegative function

$\int f < \infty$ integrable

given by a "well-guaranteed" function oracle

$x \to \boxed{} \to f(x)$ $\varepsilon > 0$

<u>Output</u>: F s.t. $(1-\varepsilon)\int f \leq F \leq (1+\varepsilon)\int f$.

Eg. What is the Gaussian measure of a polytope?

Need to make assumptions even to be able to approximate.   K - convex

f - logconcave (we'll see later)

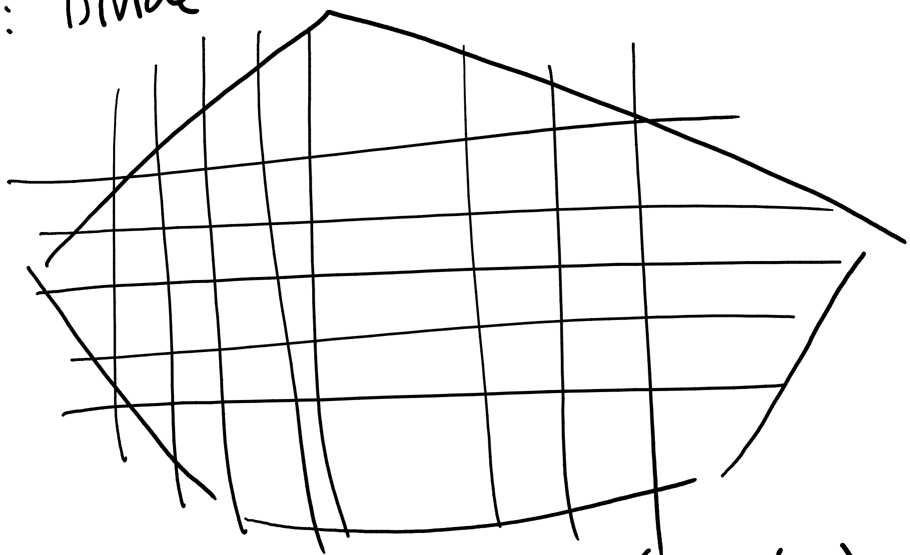
(common generalization of χ_K and $e^{-\frac{\|x\|^2}{2}}$)

<u>Problem</u>. Approximate volume of a given convex body

[illegible] oracle   arithmetic complexity.

__Complexity__: # calls to oracle, arithmetic complexity.

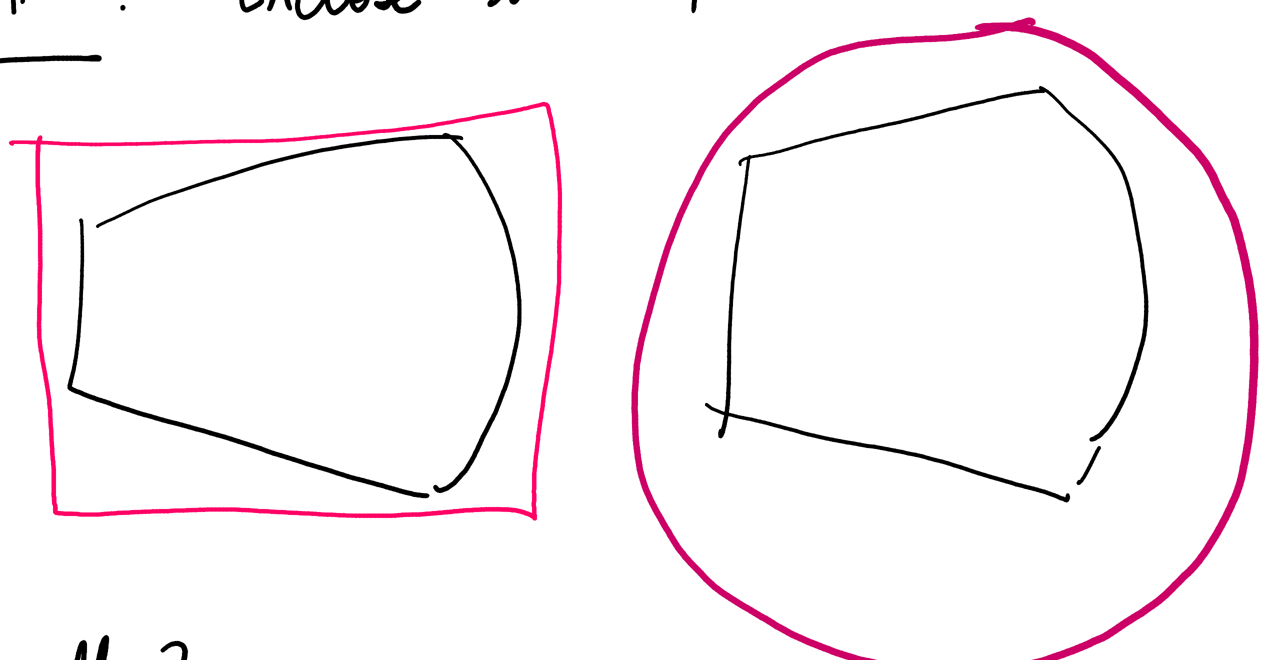
__Attempt #1__: Divide-and-conquer



For small enough cubes, $\mathrm{vol}(\bigcup \text{cubes}) \simeq \mathrm{vol}(K)$.

Difficulty?

__Attempt #2__: Enclose in simple shape



Difficulty?

Difficulty?

Attempt #3. Divide polytope P into simplices.

Difficulty?

Thm. [E, BF] For any deterministic algorithm that uses at most n^a oracle calls to estimate A, B s.t. $A \le vol(K) \le B$, $\exists K$ s.t.

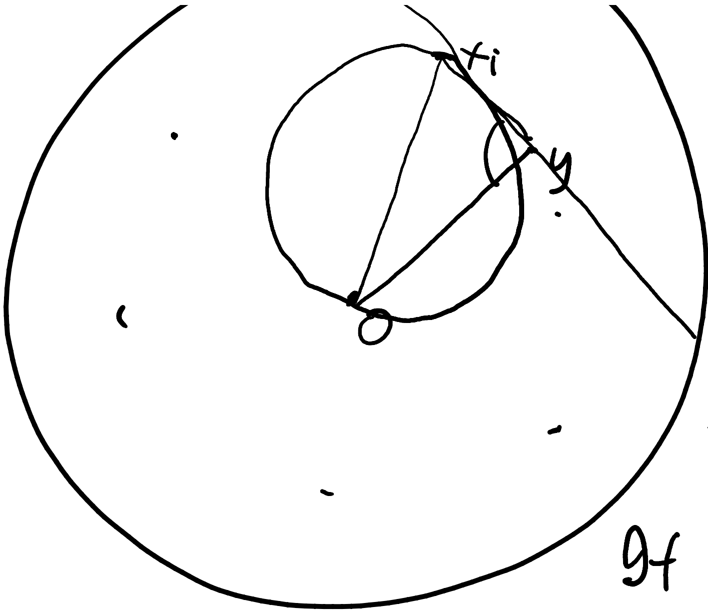
$$\frac{B}{A} > \left(\frac{cn}{a \log n}\right)^{\frac{n}{2}}$$

Thm. [BF] $\exists c, c'$ s.t. 2^{cn} deterministic approximation is not possible in $2^{c'n}$ time.

Why difficult?

$x_1 \ldots x_m$

Lemma 1. $conv\{x_1 \ldots x_m\} \subseteq \bigcup B_i$



1. $\subseteq \bigcup B_i$

2. $\mathrm{Vol}(\bigcup B_i) \le \frac{m}{2^n}$.

Pf. Take any $y \in \mathrm{conv}\{x_1 \ldots x_m\}$

If $y \notin B_i$,

then angle at $y > \frac{\pi}{2}$

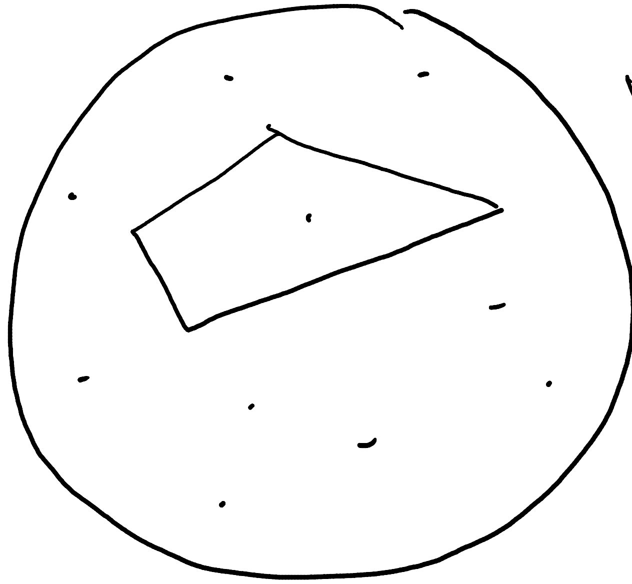
$\Rightarrow \quad y^T x_i < y^T y$

But this holds for all x_i !

So y can be separated from all x_i

and their convex hull.

Randomization is essential.



$\mathrm{Vol}(K) = \mathrm{Vol}(B) \cdot \Pr(X \in K)$

DFK algorithm. $\quad K_i = K \cap 2^{i/n} B$

LV algorithm.

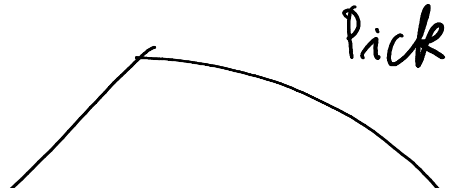
CV algorithm.

Volume / integration $\longrightarrow$ Sampling.

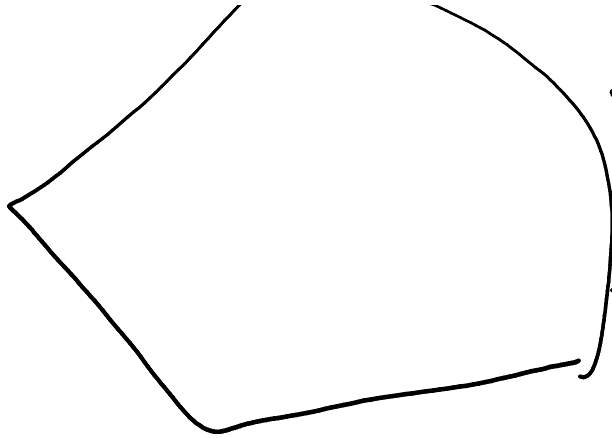
$$\mathrm{Vol}(K) = \mathrm{Vol}(B_n) \cdot \frac{\mathrm{Vol}(K_1)}{\mathrm{Vol}(B_n)} \cdot \;\cdots\; \frac{\mathrm{Vol}(K_{i+1})}{\mathrm{Vol}(K_i)} \cdots$$

Estimate each ratio using random samples.

Pick $X \sim K_{i+1}$



$Y \quad \{1 \quad X \in K_i$



$$Y = \begin{cases} 1 & X \in K_i \\ 0 & \text{o.w.} \end{cases}$$

$$E(Y) = \frac{Vol(K_i)}{Vol(K_{i+1})}$$

<u>Lema</u> $Vol(K_{i+1}) \leq 2\, Vol(K_i)$

<u>Pf.</u>

$$\begin{aligned} K_{i+1} &= 2^{\frac{i+1}{n}} B_n \cap K \\ &\subseteq 2^{\frac{1}{n}} \left(2^{i/n} B_n \cap K\right) \\ &\subseteq 2^{\frac{1}{n}} K_i \end{aligned}$$

$$W_i = \frac{1}{k} \sum_{i=1}^{k} Y_i$$

$$V = \prod_{i=1}^{m} W_i$$

$$\begin{aligned} Var(V) &= \frac{E(V^2)}{E(V)^2} - 1 \\ &= \prod_i \frac{E(W_i^2)}{E(W_i)^2} - 1 \end{aligned}$$

$$= \prod_i \left(1 + \frac{\mathrm{Var}(W_i)}{E(W_i)^2}\right) - 1$$

Suffice to set $\frac{\mathrm{Var}(W_i)}{E(W_i)^2} \leq \frac{c \cdot \varepsilon^2}{m}$

to get $\mathrm{Var}(V) \leq \varepsilon^2$.

$\Rightarrow K = O\left(\frac{m}{\varepsilon^2}\right)$ in each phase

Total # samples $= O\left(\frac{m^2}{\varepsilon^2}\right) = O\left(\frac{n^2 \log^2 R}{\varepsilon^2}\right)$

How to sample?

1. Grid walk
2. Ball walk
3. Hit-and-Run

(general).

4. Dikin, Geodesic, Hamiltonian (Polytopes)

Stationary distribution is uniform