Kruskal's MST algorithm:

Greedy approach:

For input $G=(V,E)$
Sort the edges by ↑ weight
Let $X=\emptyset$.
Go thru the edges in ↑ order of weight
 For edge $e=(y,z)$
  If $X \cup e$ is acyclic
   then add e into X.

How do we test if adding edge $e=(y,z)$ to X will create a cycle?

Check if in the graph (V,X),
 are y & z in the same component?
 if in the same component, then adding e creates a cycle
 if in diff. components then adding it is OK.

To check if y & z are in the same component, we'll use a new data structure called <u>Union-Find</u>.

First, why is Kruskal's algorithm correct?

Suppose by induction that a set X is part of some MST T.

Consider the next edge $e=(y,z)$ that we add to X.

We want to show that $X \cup e$ is part of some MST.

Let G' be the graph on edges X, so $G'=(V,X)$.

In G', let $c(y)$ be the component containing vertex y, and $c(z)$ be the component for z.

We're adding e to X so it must be the case that $c(y) \neq c(z)$.

<u>Claim:</u> e is the min weight edge from $c(y)$ to the rest of the graph.

Why? Suppose there's an edge $e' = (a,b)$ where $a \in c(y)$, $b \notin c(y)$ & $w(e') < w(e)$.

Then Kruskal's alg. considers e' before e, and it would have $c(a) \neq c(b)$ so it would add e'. Then ~~z~~ b would be in $c(y)$. $\Rightarrow\Leftarrow$

Let $S = c(y)$.

Since e is the min weight edge of E crossing $S \leftrightarrow \overline{S}$

& since no edge of X crosses $S \leftrightarrow \overline{S}$,

Then by the cut property,

$X \cup e \subset T'$ for a MST T'.

Union-find data structure:

- collection of sets — each set corresponds to a component in the graph (V, X)
- each set has a unique name — there is a specific "root" vertex in each component, and the set's name is the root vertex.

Operations:

Makeset(x): create a new set just containing x

Find(x): What is the name of the set containing x?

Union(x, y): Merge the sets containing x & y.

$O(1)$ time per Makeset

$O(\log n)$ per Find, and per Union

To check if y & z are in the same or different components, just check if $\text{find}(y) = \text{find}(z)$?

When adding edge $e = (y, z)$ to X, then to merge components $c(y)$ & $c(z)$, do $\text{Union}(y, z)$.

Kruskal(G, w):

input: connected, undirected $G = (V, E)$ with edge weights $w(e) > 0$ for $e \in E$

output: MST defined by $X \subset E$.

for all $z \in V$, Makeset(z)

$X = \emptyset$

Sort E by $\uparrow$ weight

For edge $e = (y, z)$: (go thru edges by $\uparrow$ weight)

if $\text{Find}(y) \neq \text{Find}(z)$

then add e to X

Union(y, z)

Return(X)

Running time: $n = |V|$, $m = |E|$.

Sorting $E \Rightarrow O(m \log n)$ time.

n makesets $\Rightarrow O(n)$ time

$O(m)$ Finds & n Unions $\Rightarrow O(m \log n)$ time.

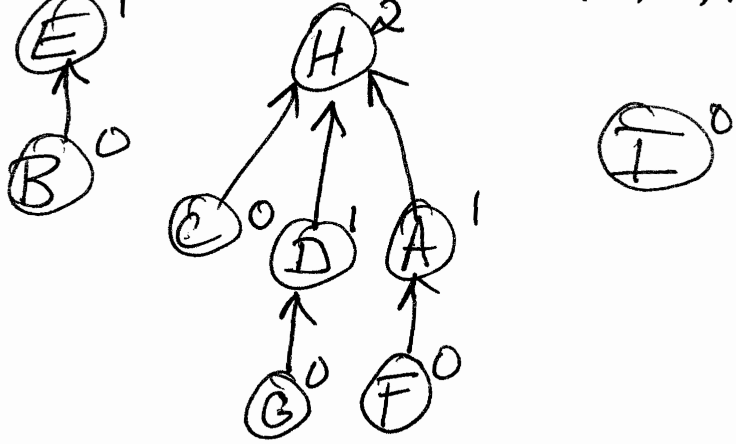
Total time: $O(m \log n)$ time.

Union-find data structure:

Each set is a directed tree:

- edges point up to the root.
- name of the set is the root.

Example: $\{B, E\}, \{A, C, D, F, G, H\}, \{I\}$



Each node x has 2 values:

1) $\pi(x)$ = parent of x

if x is the root, then $\pi(x) = x$

2) $\text{rank}(x)$ = height of subtree below x.

Makeset(x):

$\pi(x) = x$

$\text{rank}(x) = 0$

Find(x):

While $x \neq \pi(x)$ do:

$x = \pi(x)$

Return(x).

To merge sets containing x & y,
Point root of one to root of other

Key: to minimize depth,
Point root with smaller depth to larger.

So root with smaller rank points to larger rank.

Union(x,y):

$r_x = \text{Find}(x)$
$r_y = \text{Find}(y)$
If $\text{rank}(r_x) > \text{rank}(r_y)$
then $\pi(r_y) = r_x$
If $\text{rank}(r_y) > \text{rank}(r_x)$
then $\pi(r_x) = r_y$
If $\text{rank}(r_x) = \text{rank}(r_y)$
then $\left[\begin{array}{l} \pi(r_x) = r_y \\ \text{rank}(r_x)\text{++} \end{array}\right.$

Key claim: max depth is $\leq \log n$

Hence, find & union take $O(\log n)$ time.

Claim 2: Root of rank k has $\geq 2^k$ nodes in its subtree (including itself).

From claim 2: Let l be # of nodes of rank k.

Then $(l)(2^k) \leq n$

so $l \leq n/2^k$

let $k = \log_2 n + 1$

Then, $l \leq \frac{1}{2} < 1$ so there are 0 nodes of rank $> \log_2 n$.

Proof of claim 2: by induction on k.

Base case: $k=0$: it includes itself, so $2^0 = 1$ ✓

Assume true for rank $<k$.

Consider node of rank k.

It was formed by union of 2 nodes of rank $k-1$.

By induction, each had $\geq 2^{k-1}$ in their subtree.

So now there are $\geq 2 \times 2^{k-1} = 2^k$ in the subtree. ✓